Advances in the Theory of Nonlinear Analysis and its Applications 2 (2018) No. 1, 1–10
DOI:10.31197\atnaa.379280
Available online at www.atnaa.org
Review Article

Advances in the Theory of Nonlinear Analysis and its Applications
ISSN: 2587-2648
Peer-Reviewed Scientific Journal

Some problems in the fixed point theory

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Abstract

In this paper we present some of my favorite problems, all the time open, in the fixed point theory. These problems are in connection with the following two:

• Which properties have the fixed point equations for which an iterative algorithm is convergent ?

• Let us have a fixed point theorem, T, and an operator f (single or multivalued) which does not satisfy the conditions in T. In which conditions the operator f has an invariant subset Y such that the restriction of f to Y, $f\big|_Y$, satisfies the conditions of T ?

Keywords: ordered set, L-space, metric space, Banach space, Picard operator, weakly Picard operator, fixed point, fixed point structure, iterative algorithm, retraction-displacement condition, well-posedness of fixed point problem, Ostrowski property, global asymptotic stability, open problem, conjecture.
2010 MSC: 47H10, 54C60, 65J15.

1. Introduction

In this paper we present some problems, all the time open problems, in the fixed point theory. These problems are in connection with the following two research directions:

(I) Which properties have the fixed point equations for which an iterative algorithm is convergent ?

(II) Let us have a fixed point theorem, T, and an operator f (single or multivalued) which does not satisfy the conditions in the theorem T. In which conditions the operator f has an invariant subset Y such that the restriction of f to Y, $f\big|_Y$, satisfies the conditions of T ?

Throughout this paper, the standard notations and terminology are used. See for example, [33], [37] and [49]. For the basic fixed point theorems, see: [13], [19], [3], [9], [49] and [55].

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Received December 12, 2017, Accepted December 28, 2017, Online January 07, 2018.

2. Picard and weakly Picard operators

Let $(X, \to)$ be an L-space ((X,τ)-topological space, $\xrightarrow{\tau}$; (X,d)-metric space, $\xrightarrow{d}$; $(X,\|\cdot\|)$-normed space, $\xrightarrow{\|\cdot\|}$, $\rightharpoonup$; ...) and $f : X \to X$ be an operator.

By definition, f is a weakly Picard operator if the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converges for all $x \in X$ at its limit (which may depend on x) is a fixed point of f. If f is a weakly Picard operator, then we consider the operator $f^\infty : X \to X$, defined by, $f^\infty(x) := \lim_{n\to\infty} f^n(x)$.

We remark that the operator f^∞ is a set retraction on the fixed point set of f, F_f.

If f is a weakly Picard operator and $F_f = \{x^*\}$, then by definition f is called Picard operator. If f is a Picard operator, we have that,

$$F_f = F_{f^n} = \{x^*\}, \text{ for all } n \in \mathbb{N}^*$$

and if f is a weakly Picard operator, then,

$$F_f = F_{f^n} \neq \emptyset, \text{ for all } n \in \mathbb{N}^*.$$

In the case of a metric space and of a contraction we have the following result.

Theorem 2.1 (see [47]). *Let (X,d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have:*

(i) *f is a Picard operator ($F_f = \{x^*\}$).*

(ii) *$d(x,x^*) \leq \psi(d(x,f(x)))$, for all $x \in X$, where $\psi(t) = \frac{t}{1-l}$, $t \geq 0$.*

(iii) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.

(iv) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ as $n \to \infty$.

From this result, the following problem rises:

Problem 2.2. Let (X,d) be a complete metric space and $f : X \to X$ be an operator. Which metric conditions on f imply a similar conclusion as that of Theorem 2.1 ?

Let us consider another result:

Theorem 2.3 (see [48]). *Let (X,d) be a complete metric space and $f : X \to X$ be an operator. We suppose that:*

(1) *There exists $l \in]0,1[$ such that,*

$$d(f^2(x), f(x)) \leq l d(x, f(x)), \text{ for all } x \in X,$$

i.e., f is a graphic contraction.

(2) *$\lim_{n\to\infty} f(f^n(x)) = f(\lim_{n\to\infty} f^n(x))$, for all $x \in X$.*

Then we have:

(*i*) *f is a weakly Picard operator.*

(*ii*) $d(x, f^{\infty}(x)) \leq \frac{1}{1-l} d(x, f(x))$, *for all* $x \in X$.

(*iii*) *For* $x^* \in F_f$, *let* $X_{x^*} := \{x \in X \mid f^n(x) \to x^* \text{ as } n \to \infty\}$. *Let* $\{y_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_{x^*} *such that*

$$d(y_n, f(y_n)) \to 0 \text{ as } n \to \infty.$$

Then, $y_n \to x^*$ *as* $n \to \infty$.

(*iv*) *Let* $\{y_n\}_{n\in\mathbb{N}}$ *be a sequence in* X_{x^*}, $x^* \in F_f$. *If* $l < \frac{1}{3}$ *and*

$$d(y_{n+1}, f(y_n)) \to 0 \text{ as } n \to \infty,$$

then, $y_n \to x^*$ *as* $n \to \infty$.

This result suggests the following problem:

Problem 2.4 (see [48]). Which metric conditions imposed on an operator f imply a similar conclusion as that in Theorem 2.3 ?

For a better understanding of the above problems, let us consider the following considerations:

(*a*) A weakly Picard operator $f : (X, d) \to (X, d)$ satisfies a retraction-displacement condition (see [8]) if there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(0) = 0$ and continuous in 0, such that

$$d(x, f^{\infty}(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

This condition is useful in studying the data dependence of the fixed point, and of Ulam stability of the fixed point equations (see [44]).

So, conclusions (*ii*) in Theorems 2.1 and 2.3 are retraction-displacement conditions for the operator f.

(*b*) Conclusions (*iii*) in Theorems 2.1 and 2.3 can be formulated as follows: The fixed point problem for the operator f is well posed.

(*c*) Conclusions (*iv*) in Theorems 2.1 and 2.3 can be formulated as follows: The operator f has the Ostrowski property.

Problem 2.5. To study similar problems in the case of multivalued operators.

References for Problems 2.2 - 2.5: [47], [48], [39], [50], [52], [8], [28], [31], [32], [49], [51], [56], [57], [54], ...

Problem 2.6. To study similar problems in the case of a convergent iterative algorithm.

References: [42], [27], [7], [6], [25], [26], ...

3. Conjecture on global asymptotic stability

Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. A fixed point x^* of f is by definition globally asymptotically stable if f is a Picard operator, i.e., $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

In 1976, J.P. LaSalle presented (see [20]) the following conjecture:

Conjecture 1 (LaSalle's Conjecture). *Let* $f : \mathbb{R}^m \to \mathbb{R}^m$ *be such that:*

(*i*) *there exists* $x^* \in \mathbb{R}^m$ *with* $f(x^*) = x^*$;

(*ii*) *$f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$;*

(*iii*) *the spectral radius of the differential of f at x, $\rho(df(x)) < 1$, for all $x \in \mathbb{R}^m$.*

Then, x^* is globally asymptotically stable.

Papers on this conjecture were given by (see [46]): A. Cima - A. Gasull - F. Mañosas (1995, 1999, 2001, 2011, 2014), G. Meisters (1996), A.G. Aksoy - M. Martelli (2001), A. Castañeda - V. Guiñez (2012), D. Cheban (2014), ... The results are as follow:

(*a*) counterexamples to LaSalle Conjecture;

(*b*) classes of functions for which LaSalle Conjecture is a theorem;

(*c*) to study the dynamic generated by a function $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, with $\rho(df(x)) < 1$, for all $x \in \mathbb{R}^m$.

We have the following remark: Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. The following statements are equivalent:

(*i*) f is a Picard operator;

(*ii*) for all $k \in \mathbb{N}^*$, f^k is a Picard operator;

(*iii*) there exists $k \in \mathbb{N}^*$ such that f^k is a Picard operator.

Starting from this general remark, in [46] the following conjecture is presented.

Problem 3.1 (a conjecture). Let X be a real Banach space, $\Omega \subset X$ be an open, convex subset and $f : \Omega \to \Omega$ be an operator. We suppose that:

(*i*) $f \in C^1(\Omega, X)$;

(*ii*) $\rho(df^k(x)) < 1$, for all $x \in \Omega$ and all $k \in \mathbb{N}^*$;

(*iii*) $F_f \neq \emptyset$.

Then, f is a Picard operator.

In connection with the above conjecture the following problems arise:

Problem 3.2. In which conditions we have that:

$$\rho(df(x)) < 1, \text{ for all } x \in \Omega \ \Rightarrow \ \rho(df^k(x)) < 1, \text{ for all } x \in \Omega \text{ and all } k \in \mathbb{N}^*?$$

Problem 3.3. In which conditions we have that:

$$\rho(df(x)) < 1, \text{ for all } x \in \Omega \ \Rightarrow \ f \text{ is nonexpansive with respect to an equivalent norm on } X?$$

We remember that if $(X, \|\cdot\|)$ is a complex Banach space and $f : X \to X$ is a bounded linear operator with the spectrum $\sigma(f)$, then (see [17], [5], [14], [4], ...)

$$\rho(f) = \sup_{\lambda \in \sigma(f)} |\lambda| = \lim_{n \to \infty} \|f^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}^*} \|f^n\|^{\frac{1}{n}} = \inf_{|\cdot| \sim \|\cdot\|} |f|.$$

If X is a real Banach space and $f : X \to X$ is a bounded linear operator, $X_{\mathbb{C}}$ the complexification of X, $f_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ the complexification of f, then by definition, $\rho(f) := \rho(f_{\mathbb{C}})$.

References: [46], [20], [4], [25], [26], ...

4. Nonexpansive operators and graphic contractions

Problem 4.1. Let $(X, \|\cdot\|)$ be a (real or complex) Banach space. Which nonexpansive operators $f : X \to X$ are graphic contractions ?

Commentaries: If f is a graphic contraction then $\inf_{x\in X} \|x - f(x)\| = 0$. If $\Omega \subset X$ is an invariant subset of f and f is a graphic contraction then, $\inf_{x\in X} \|x - f(x)\| = 0$. On the other hand, in the case of nonexpansive operators we have the following Goebel-Karlovitz Lemma (see [12]): Let $\Omega \subset X$ be a convex, closed and bounded subset. Let $D \subset \Omega$ be a weakly compact, convex, minimal invariant set for a nonexpansive operator $f : \Omega \to \Omega$. If for a sequence $\{x_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} \|x_n - f(x_n)\| = 0$, then for any $z \in D$, we have that, $\lim_{n\to\infty} \|z - x_n\| = diam(D)$.

So, the above problem is a hard one.

Problem 4.2. Let X be an ordered Banach space. Which increasing, linear and nonexpansive operators $f : X \to X$ are graphic contractions ?

Problem 4.3. Let X be a Banach space. Which multivalued nonexpansive operators $T : X \to P(X)$ are graphic contractions ?

References: [36], [40], [43], [45], [1], [2], [10], [16], [19], [18], [30], [39], [49], ...

5. Abstract and concrete Gronwall lemmas

Let $(X, \to, \leq)$ be an ordered L-space and $f : X \to X$ be an operator. The following results are well known (see [38]:

Lemma 5.1 (Abstract Gronwall Lemma for Picard operators). *We suppose that:*

(i) *f is a Picard operator ($F_f = \{x^*\}$);*

(ii) *f is an increasing operator.*

Then we have that:

(a) *$x \in X$, $x \leq f(x) \Rightarrow x \leq x^*$;*

(b) *$x \in X$, $x \geq f(x) \Rightarrow x \geq x^*$.*

Lemma 5.2 (Abstract Gronwall Lemma for weakly Picard operators). *We suppose that:*

(i) *f is a weakly Picard operator;*

(ii) *f is an increasing operator*

Then we have that:

(a) *$x \in X$, $x \leq f(x) \Rightarrow x \leq f^\infty(x)$;*

(b) *$x \in X$, $x \geq f(x) \Rightarrow x \geq f^\infty(x)$.*

The above abstract Gronwall lemmas are very usefully for giving some concrete Gronwall lemmas. On the other hand a large number of concrete Gronwall lemmas are obtained by direct proofs. The following problems are arising:

Problem 5.3. In which Gronwall lemmas the upper bounds are fixed points of the corresponding operator ?

Problem 5.4. If there are found solutions for the Problem 5.3, which of them are consequences of some abstract Gronwall lemmas ?

References: [38], [35], [21], [11], [22], [23], [33], [39], [49], ...

6. Invariant subsets with fixed point property

For a rigorous formulation of a problem (II), from Introduction, we recall a few basic notions and examples of the fixed point structure theory (see [37]).

Let $\mathcal{C}$ be a class of structured sets (ordered sets, ordered linear spaces, topological spaces, metric spaces, Hilbert spaces, Banach spaces, ordered Banach spaces, generalized metric spaces, ...). Let Set^* be the class of nonempty sets and if X is a nonempty set, then, $P(X) := \{Y \subset X \mid Y \neq \emptyset\}$. We also shall use the following notations:

$P(\mathcal{C}) := \{U \in P(X) \mid X \in \mathcal{C}\}$,
$\mathbb{M}(U,V) := \{f : U \to V \mid f \text{ is an operator}\}$,
$\mathbb{M}(U) := \mathbb{M}(U,U)$,
$S : \mathcal{C} \multimap Set^*$, $X \mapsto S(X) \subset P(X)$,
$M : D_M \subset P(\mathcal{C}) \times P(\mathcal{C}) \multimap \mathbb{M}(P(\mathcal{C}), P(\mathcal{C}))$, $(U,V) \mapsto M(U,V) \subset \mathbb{M}(U,V)$

By a fixed point structure (f.p.s.) on $X \subset \mathcal{C}$ we understand a triple $(X, S(X), M)$ with the following properties:

(i) $U \in S(X) \Rightarrow (U,U) \in D_M$;

(ii) $U \in S(X)$, $f \in M(U) \Rightarrow F_f \neq \emptyset$;

(iii) M is such that:

$$(Y,Y) \in D_M,\ Z \in P(Y),\ (Z,Z) \in D_M\ \Rightarrow M(Z) \supset \{f\big|_Z \mid f \in M(Y)\}.$$

Here are some examples of f.p.s.

Example 6.1 (The f.p.s. of progressive operators)**.** Let $\mathcal{C}$ be the class of partially ordered sets. For $(X, \leq) \in \mathcal{C}$, let

$$S(X) := \{Y \in P(X) \mid (Y, \leq) \text{ has at least a maximal element}\}$$

and

$$M(Y) := \{f : Y \to Y \mid x \leq f(x), \text{ for all } x \in Y\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Example 6.2 (The Tarski's f.p.s.)**.** Let $\mathcal{C}$ be the class of partially ordered sets. For $(X, \leq) \in \mathcal{C}$, let

$$S(X) := \{Y \in P(X) \mid (Y, \leq) \text{ is a complete lattice}\}$$

and

$$M(Y) := \{f : Y \to Y \mid f \text{ is an increasing operator}\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Example 6.3 (The f.p.s. of contractions)**.** Let $\mathcal{C}$ be the class of complete metric spaces. Let

$$S(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$$

and

$$M(Y) := \{f : Y \to Y \mid f \text{ is a contraction}\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Example 6.4 (The f.p.s. of Schauder)**.** Let $\mathcal{C}$ be the class of Banach spaces. Let

$$S(X) := \{Y \in P(X) \mid Y \text{ is compact and convex}\}$$

and

$$M(Y) := \{f : Y \to Y \mid f \text{ is continuous}\}.$$

Then, $(X, S(X), M)$ is a f.p.s.

Now, our problem (II) takes the following form:

Problem 6.5. Let $(X, S(X), M)$ be a f.p.s. on $X \in \mathcal{C}$ and $f : A \to A$ be an operator with $A \subset X$. In which conditions there exists $Y \subset A$ such that

(*a*) $Y \in S(X)$;

(*b*) $f(Y) \subset Y$;

(*c*) $f\big|_Y \in M(Y)$?

We have a similar problem in the case of multivalued operators.
References: [37], [41], [29], [49], ...

7. Strict fixed point problems

Let X be a nonempty set and $T : X \to P(X)$ be a multivalued operator. Let $F_T := \{x \in X \mid x \in T(x)\}$ be the set of fixed point of T and $(SF)_T := \{x \in X \mid T(x) = \{x\}\}$ be the strict fixed point set of T.

We have the following result (see [33], p.87):

Let (X, d) be a metric space and $T : X \to P(X)$ be a multivalued l-contraction. If, $(SF)_T \neq \emptyset$, then,

$$F_T = (SF)_T = \{x^*\}.$$

The following problem is arising:

Problem 7.1. For which multivalued generalized contractions we have that

$$(SF)_T \neq \emptyset \;\Rightarrow\; F_T = (SF)_T = \{x^*\} \text{ ?}$$

Problem 7.2. Let $(X, S(X), M^\circ)$ be a multivalued fixed point structure (see [37]) on $X \in \mathcal{C}$. Let $Y \in S(X)$ and $T \in M^\circ(Y)$. In which conditions we have that

$$F_T = (SF)_T?$$

Commentaries:

(1) Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that:

  (*a*) $F_f = F_g$;

  (*b*) $x \leq f(x) \leq g(x)$, for all $x \in \mathbb{R}$.

  Let $T : \mathbb{R} \to P(\mathbb{R})$ be defined by,

$$T(x) := \{tf(x) + (1-t)g(x) \mid t \in [0, 1]\}.$$

  Then we have that, $F_T = (SF)_T$.

(2) Let (X,d) be a metric space, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ be a partition of X, and for each $\lambda \in \Lambda$, $T_\lambda : X_\lambda \to P(X_\lambda)$ be a multivalued contraction with respect to the Pompeiu-Hausdorff functional. We suppose that, $(SF)_{T_\lambda} \neq \emptyset$, for all $\lambda \in \Lambda$.

Let $T : X \to P(X)$ be defined by, $T(x) = T_\lambda(x)$, if $x \in X_\lambda$, $\lambda \in \Lambda$.

It is clear that, $F_T = (SF)_T \neq \emptyset$.

(3) Let $(X, S(X), M)$ be a fixed point structure of progressive operators on a partially ordered set $(X, \leq)$. Let $Y \in S(X)$ and $f, g \in M(Y)$. We suppose that:

(a) $f(x) \leq g(x)$, for all $x \in Y$;

(b) $x < f(x)$, for each nonmaximal element $x \in Y$.

Let $T : Y \to P(Y)$ be a multivalued operator defined by,

$$T(x) := \{y \in Y \mid f(x) \leq y \leq g(x)\}.$$

Then, $F_T = (SF)_T \neq \emptyset$.

References: [34], [53], [28], [49], [31], ...

8. Commutative pairs of operators with coincidence property

Problem 8.1. Which are the f.p.s. $(X, S(X), M)$, $X \in \mathcal{C}$, with the following property:

$$Y \in S(X),\ f, g \in M(Y),\ f \circ g = g \circ f \ \Rightarrow \ \text{there exists } x \in Y \text{ such that } f(x) = g(x)?$$

Commentaries:

(1) In the case of Tarski's fixed point structure we have that, $F_f \cap F_g \neq \emptyset$.

(2) In the case of Schauder's fixed point structure, the Problem 8.1 takes the following form:

Conjecture 2 (Horn's Conjecture)**.** *Let X be a Banach space, $Y \subset X$, compact and convex subset and $f, g : Y \to Y$ be two continuous operators. If $f \circ g = g \circ f$, then there exists $x \in Y$ such that $f(x) = g(x)$.*

(3) The Horn's Conjecture includes:

Conjecture 3 (Schauder-Browder-Nussbaum Conjecture)**.** *Let X be a Banach space, $Y \subset X$ be a bounded, closed and convex subset and $f : Y \to Y$ be a continuous operator. If there exists $n_0 \in \mathbb{N}^*$ such that f^{n_0} is compact, then $F_f \neq \emptyset$.*

References: [37], [41], [15], [24], [18], [49], ...

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