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Fixed point theorem for a kind of Ćirić type contractions in complete metric spaces

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Abstract

We prove a fixed point theorem for a kind of Ćirić type contractions in complete metric spaces. In order to demonstrate the assumption of the fixed point theorem, we give an example. We also clarify the mathematical structure of some fixed point theorem proved by Mınak-Helvacı-Altun and Wardowski-Dung independently.

Keywords: Fixed point, F-contraction, contractive condition.
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1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In 2012, Wardowski in [10] introduced the concept of F-contraction and proved the following fixed point theorem.

Theorem 1.1 (Theorem 2.1 in Wardowski [10]). *Let (X, d) be a complete metric space and let T be a F-contraction on X, that is, there exist a function F from $(0, \infty)$ into $\mathbb{R}$ and real numbers $\tau \in (0, \infty)$ and $k \in (0, 1)$ satisfying the following*:

(F1) *F is strictly increasing.*
(F2) *For any sequence $\{\alpha_n\}$ of positive numbers,* $\lim_n \alpha_n = 0 \iff \lim_n F(\alpha_n) = -\infty$.
(F3) $\lim[t^k F(t) : t \to +0] = 0$.
(F4) $\tau + F \circ d(Tx, Ty) \leq F \circ d(x, y)$ *for any $x, y \in X$ with $Tx \neq Ty$.*

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Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

In 2014, Mınak, Helvacı and Altun in [6] and Wardowski and Dung in [11] independently proved the following fixed point theorem.

Theorem 1.2 (Theorem 2.2 in [6], Theorem 2.4 in [11])**.** *Let (X, d) be a complete metric space and let T be a mapping on X. Assume that there exist a function F from $(0, \infty)$ into $\mathbb{R}$ and real numbers $\tau \in (0, \infty)$ and $k \in (0, 1)$ satisfying* (F1)–(F3) *and the following*:

(F5) $\tau + F \circ d(Tx, Ty) \le F \circ L(x, y)$ *for any $x, y \in X$ with $Tx \ne Ty$, where L is defined by*

$$L(x, y) = \max\left\{d(x, y), \frac{d(x, Ty) + d(Tx, y)}{2}, d(x, Tx), d(y, Ty)\right\}. \tag{1.1}$$

Assume also either of the following:

(F6) *T is continuous.*

(F7) *F is continuous.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

We assume (F6) or (F7) additionally. So, we note that Theorem 1.2 is not a generalization of Theorem 1.1. Also F appears in both sides of (F5). So, we do not understand the mathematical structure of Theorem 1.2 easily.

Motivated by the above, in this paper, we clarify the mathematical structure of Theorem 1.2. Indeed, in the case of (F6), we can prove Theorem 1.2 by using the known result (Theorem 4.1). Also we can weaken the assumption of (F7) (see Theorem 2.1). In both cases, we do not use F. Finally we give an example (Example 5.1), which implies that we cannot generalize Theorem 1.1 with using L. Also, Example 5.1 tells that the assumption of the new fixed point theorem (Theorem 2.1) is reasonably weak.

2. Main Result

In this section, we prove the following fixed point theorem.

Theorem 2.1. *Let (X, d) be a complete metric space and let T be a mapping on X. Define a function L from $X \times X$ into $[0, \infty)$ by (1.1). Assume that there exists a function φ from $[0, \infty)$ into itself satisfying the following*:

(i) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*

(ii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon < t < \varepsilon + \delta \quad \textit{implies} \quad \varphi(t) \le \varepsilon.$$

(iii) *$d(Tx, Ty) \le \varphi \circ L(x, y)$.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

Remark 2.2.

- *Define a subset Q of $[0, \infty)^2$ by*

$$Q = \big\{\big(L(x, y), d(Tx, Ty)\big) : x, y \in X\big\}. \tag{2.1}$$

Then Q satisfies Condition $C(0, 1, 0)$ (see Section 3).

- *Since we do not assume the nondecreasingness of φ, we cannot prove this theorem by using Theorem 5 in [9].*

Proof of Theorem 2.1. Since $L(x,y) = 0$ implies $d(Tx,Ty) = 0$, without loss of generality, we may assume $\varphi(0) = 0$. By (i), we can easily prove that (ii) is equivalent to the following:

(ii') For any $\varepsilon > 0$, there exists $\delta > 0$ such that $t < \varepsilon + \delta$ implies $\varphi(t) \le \varepsilon$.

We will show the following:

$$x \ne y \quad \Rightarrow \quad d(Tx,Ty) < L(x,y), \tag{2.2}$$
$$d(Tx,Ty) \le L(x,y). \tag{2.3}$$

Indeed, if $x \ne y$ holds, then $L(x,y) > 0$ holds. We have by (i) and (iii)

$$d(Tx,Ty) \le \varphi \circ L(x,y) < L(x,y).$$

Thus (2.2) holds. If $x = y$ holds, then we have

$$d(Tx,Ty) = 0 \le L(x,y).$$

Combining this with (2.2), we obtain (2.3).

We next show the following:

$$x \ne Tx \quad \Rightarrow \quad d(Tx,T^2x) < d(x,Tx), \tag{2.4}$$
$$d(Tx,T^2x) \le d(x,Tx), \tag{2.5}$$
$$L(x,Tx) = d(x,Tx). \tag{2.6}$$

Indeed we have

$$\begin{aligned} L(x,Tx) &= \max\Big\{d(x,Tx), \frac{d(x,T^2x) + d(Tx,Tx)}{2}, d(x,Tx), d(Tx,T^2x)\Big\} \\ &= \max\Big\{d(x,Tx), \frac{d(x,T^2x)}{2}, d(Tx,T^2x)\Big\} \\ &= \max\Big\{d(x,Tx), \frac{d(x,Tx) + d(Tx,T^2x)}{2}, d(Tx,T^2x)\Big\} \\ &= \max\{d(x,Tx), d(Tx,T^2x)\}. \end{aligned}$$

If $x \ne Tx$ holds, then we have by (2.2)

$$d(Tx,T^2x) < L(x,Tx) = \max\{d(x,Tx), d(Tx,T^2x)\}.$$

So we obtain (2.4). Using (2.4), we can prove (2.5) and (2.6).

Fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$ for $n \in \mathbb{N}$. From (2.5), $\{d(u_n,u_{n+1})\}$ is nonincreasing. So $\{d(u_n,u_{n+1})\}$ converges to some $\varepsilon_1 \ge 0$. Arguing by contradiction, we assume $\varepsilon_1 > 0$. From (ii'), there exists $\delta_1 > 0$ satisfying the following:

- $t < \varepsilon_1 + \delta_1$ implies $\varphi(t) \le \varepsilon_1$.

From the definition of ε_1, we can choose $\nu \in \mathbb{N}$ satisfying

$$L(u_\nu, u_{\nu+1}) = d(u_\nu, u_{\nu+1}) < \varepsilon_1 + \delta_1.$$

Then we have

$$0 < \varepsilon_1 \le d(u_{\nu+1}, u_{\nu+2}) \le \varphi \circ L(u_\nu, u_{\nu+1}) \le \varepsilon_1$$

and hence by (2.4),

$$\varepsilon_1 \le d(u_{\nu+2}, u_{\nu+3}) < d(u_{\nu+1}, u_{\nu+2}) = \varepsilon_1,$$

which implies a contradiction. Therefore we obtain $\varepsilon_1 = 0$. That is, $\lim_n d(u_n,u_{n+1}) = 0$ holds. Fix $\varepsilon_2 > 0$. Then from (ii'), there exists $\delta_2 > 0$ satisfying the following:

- $t < \varepsilon_2 + 2\,\delta_2$ implies $\varphi(t) \leq \varepsilon_2$.

Let $\ell \in \mathbb{N}$ be large enough to satisfy $d(u_\ell, u_{\ell+1}) < \delta_2$. We will show

$$d(u_\ell, u_{\ell+j}) < \varepsilon_2 + \delta_2 \tag{2.7}$$

for $j \in \mathbb{N}$ by induction. It is obvious that (2.7) holds when $j = 1$. We assume that (2.7) holds for some $j \in \mathbb{N}$. Then we have by (2.5)

$$\begin{aligned}
&d(u_\ell, u_{\ell+j+1}) + d(u_{\ell+1}, u_{\ell+j})\\
&\leq d(u_\ell, u_{\ell+j}) + d(u_{\ell+j}, u_{\ell+j+1}) + d(u_{\ell+1}, u_\ell) + d(u_\ell, u_{\ell+j})\\
&< 2\,\varepsilon_2 + 4\,\delta_2.
\end{aligned}$$

Hence

$$\begin{aligned}
&L(u_\ell, u_{\ell+j})\\
&= \max\left\{ d(u_\ell, u_{\ell+j}), \frac{d(u_\ell, u_{\ell+j+1}) + d(u_{\ell+1}, u_{\ell+j})}{2}, d(u_\ell, u_{\ell+1}), d(u_{\ell+j}, u_{\ell+j+1}) \right\}\\
&< \varepsilon_2 + 2\,\delta_2
\end{aligned}$$

holds. So we have

$$d(u_{\ell+1}, u_{\ell+j+1}) \leq \varphi \circ L(u_\ell, u_{\ell+j}) \leq \varepsilon_2$$

and hence

$$d(u_\ell, u_{\ell+j+1}) \leq d(u_\ell, u_{\ell+1}) + d(u_{\ell+1}, u_{\ell+j+1}) < \delta_2 + \varepsilon_2.$$

Thus, (2.7) holds with $j := j + 1$. So, by induction, (2.7) holds for every $j \in \mathbb{N}$. Since $\varepsilon_2 > 0$ is arbitrary, we obtain

$$\lim_{n\to\infty} \sup_{m>n} d(u_n, u_m) = 0,$$

which implies that $\{u_n\}$ is Cauchy. Since X is complete, $\{u_n\}$ converges to some $z \in X$. Arguing by contradiction, we assume $\tau := d(z, Tz) > 0$. Since $\{u_n\}$ converges to z, we can choose $\mu \in \mathbb{N}$ satisfying

$$\max\{d(z, u_\mu), d(z, u_{\mu+1}), d(u_\mu, u_{\mu+1})\} < \min\{\tau - \varphi(\tau), \tau/2\}.$$

We have

$$\begin{aligned}
&L(z, u_\mu)\\
&= \max\left\{ d(z, u_\mu), \frac{d(z, u_{\mu+1}) + d(Tz, u_\mu)}{2}, d(z, Tz), d(u_\mu, u_{\mu+1}) \right\}\\
&= \max\left\{ \frac{\tau}{2}, \frac{\tau/2 + d(Tz, z) + d(z, u_\mu)}{2}, \tau, \frac{\tau}{2} \right\}\\
&= \max\left\{ \tau, \frac{\tau/2 + \tau + \tau/2}{2} \right\}\\
&= \tau.
\end{aligned}$$

Hence

$$d(Tz, u_{\mu+1}) \leq \varphi \circ L(z, u_\mu) = \varphi(\tau)$$

holds. We have

$$\tau = d(z, Tz) \leq d(z, u_{\mu+1}) + d(u_{\mu+1}, Tz) < \tau - \varphi(\tau) + \varphi(\tau) = \tau,$$

which implies a contradiction. Therefore we have shown that z is a fixed point of T.

Let $w \in X$ be a fixed point of T. Then we have

$$L(z, w) = \max\left\{ d(z, w), \frac{d(z, w) + d(z, w)}{2}, d(z, z), d(w, w) \right\} = d(z, w)$$

and hence

$$d(z, w) = d(Tz, Tw) \leq \varphi \circ L(z, w) = \varphi \circ d(z, w),$$

which implies $d(z, w) = 0$, thus, the fixed point z is unique. □

3. Contractive Conditions

In this section, we study contractive conditions.

Definition 3.1. *Let X be a nonempty set and let p and d be functions from $X \times X$ into $[0, \infty)$. Let T be a mapping on X.*

(1) *T is said to be a CJM contraction [2, 4, 5] if the following hold:*
  (1-i) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$.*
  (1-ii) *$d(Tx, Ty) < p(x, y)$ for any $x, y \in X$ with $d(Tx, Ty) > 0$.*
(2) *T is said to be of New-type [9] if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (2-i) *$\varphi(0) = 0$.*
  (2-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (2-iii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta$ implies $\varphi(t) \leq \varepsilon$.*
  (2-iv) *$d(Tx, Ty) \leq \varphi \circ p(x, y)$ for all $x, y \in X$.*
(3) *T is said to be a Browder contraction [1] if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (3-i) *φ is nondecreasing and right continuous.*
  (3-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (3-iii) *$d(Tx, Ty) \leq \varphi \circ p(x, y)$ for all $x, y \in X$.*

In order to concentrate on contractive conditions, we consider subsets of $[0, \infty)^2$, see [3]. We give definitions which are strongly connected with contractive conditions in Definition 3.1.

Definition 3.2. *Let Q be a subset of $[0, \infty)^2$.*

(1) *Q is said to be CJM if the following hold:*
  (1-i) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $u \leq \varepsilon$ holds for any $(t, u) \in Q$ with $t < \varepsilon + \delta$.*
  (1-ii) *$u < t$ holds for any $(t, u) \in Q$ with $u > 0$.*
(2) *Q is said to be of New-type if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (2-i) *$\varphi(0) = 0$.*
  (2-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (2-iii) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < t < \varepsilon + \delta$ implies $\varphi(t) \leq \varepsilon$.*
  (2-iv) *$u \leq \varphi(t)$ for all $(t, u) \in Q$.*
(3) *Q is said to be a Browder if there exists a function φ from $[0, \infty)$ into itself satisfying the following:*
  (3-i) *φ is nondecreasing and right continuous.*
  (3-ii) *$\varphi(t) < t$ for any $t \in (0, \infty)$.*
  (3-iii) *$u \leq \varphi(t)$ for all $(t, u) \in Q$.*

The following obviously holds. See also Proposition 6 in [7].

Proposition 3.3. *Let X be a nonempty set and let p and d be functions from $X \times X$ into $[0, \infty)$. Let T be a mapping on X. Define a subset Q of $[0, \infty)^2$ by*

$$Q = \big\{ \big(p(x, y), d(Tx, Ty)\big) : x, y \in X \big\}. \tag{3.1}$$

Then the following hold:

(i) *T is a CJM contraction iff Q is CJM.*
(ii) *T is of New-type iff Q is of New-type.*
(iii) *T is a Browder contraction iff Q is Browder.*

Very recently, the concept of Condition C(p, q, r) was introduced in [8]. Using this concept, we can compare contractive conditions quite easily.

Definition 3.4 ([8]). *Let Q be a subset of $[0,\infty)^2$.*

(1) *Q is said to satisfy Condition $C(0,0,0)$ if the following hold:*
  (1-i) *$u < t$ for any $(t,u) \in Q$ with $u > 0$.*
  (1-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\tau < t_n$, $\tau < u_n$ and $\lim_n t_n = \lim_n u_n = \tau$.*

(2) *Q is said to satisfy Condition $C(0,0,1)$ if the following hold:*
  (2-i) *Q satisfies Condition $C(0,0,0)$.*
  (2-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\tau < t_n$, $u_n = \tau$ and $\lim_n t_n = \tau$.*

(3) *Q is said to satisfy Condition $C(0,0,2)$ if the following hold:*
  (3-i) *Q satisfies Condition $C(0,0,0)$.*
  (3-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $\tau < t_n$, $u_n \le \tau$ and $\lim_n t_n = \lim_n u_n = \tau$.*

(4) *Q is said to satisfy Condition $C(0,1,0)$ if the following hold:*
  (4-i) *Q satisfies Condition $C(0,0,0)$.*
  (4-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $t_n = \tau$, $u_n < \tau$ and $\lim_n u_n = \tau$.*

(5) *Q is said to satisfy Condition $C(1,0,0)$ if the following hold:*
  (5-i) *Q satisfies Condition $C(0,0,0)$.*
  (5-ii) *There does not exist $\tau > 0$ and a sequence $\{(t_n, u_n)\}$ in Q satisfying $t_n < \tau$, $u_n < \tau$ and $\lim_n t_n = \lim_n u_n = \tau$.*

(6) *Let $(p,q,r) \in \{0,1\}^2 \times \{0,1,2\}$. Then Q is said to satisfy Condition $C(p,q,r)$ if Q satisfies Conditions $C(p,0,0)$, $C(0,q,0)$ and $C(0,0,r)$.*

Remark 3.5. *The expressions on the above conditions are a little different from those in [8]. Of course, both are essentially the same.*

The following was essentially proved in [8].

Proposition 3.6 ([8]). *Let Q be a subset of $[0,\infty)^2$. Then the following hold:*

(i) *Q is CJM iff Q satisfies Condition $C(0,0,0)$.*
(ii) *Q is of New-type iff Q satisfies Condition $C(0,1,0)$.*
(iii) *Q is Browder iff Q satisfies Condition $C(1,1,2)$.*

We prove the following lemma, which plays an important role in this paper.

Lemma 3.7. *Let X be a nonempty set and let p and d be functions from $X \times X$ into $[0,\infty)$. Let T be a mapping on X. Assume that there exist a nondecreasing function F from $(0,\infty)$ into $\mathbb{R}$ and a real number $\tau \in (0,\infty)$ satisfying*

$$d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F \circ d(Tx,Ty) \le F \circ p(x,y)$$

for any $x, y \in X$. Define a subset Q of $[0,\infty)^2$ by (3.1). Then the following hold:

(i) *Q satisfies Condition $C(1,0,0)$.*
(ii) *If F is right continuous, then Q satisfies Condition $C(1,0,1)$.*
(iii) *If F is left continuous, then Q satisfies Condition $C(1,1,0)$.*
(iv) *If F is continuous, then Q satisfies Condition $C(1,1,2)$.*

Remark 3.8. *(iii) and (iv) were essentially proved in [7]. See Remark below the proof of Theorem 17 in [7]. For the sake of completeness, we give a proof. We note that the proof below is much simpler than that in [7].*

Proof of Lemma 3.7. We first show (i). Let $(t, u) \in Q$ satisfy $u > 0$. Then there exist $x, y \in X$ satisfying $p(x, y) = t$ and $d(Tx, Ty) = u$. From the assumption, we have $\tau + F(u) \leq F(t)$, which implies $u < t$. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and sequences $\{t_n\}$ and $\{u_n\}$ in (υ, ∞) satisfying $(t_n, u_n) \in Q$ and $\lim_n (t_n, u_n) = (\upsilon, \upsilon)$. Then we have

$$\tau + \lim_{t \to \upsilon+0} F(t) = \tau + \lim_{n\to\infty} F(u_n) \leq \lim_{n\to\infty} F(t_n) = \lim_{t\to\upsilon+0} F(t).$$

Since $\lim_{t\to\upsilon+0} F(t) \in \mathbb{R}$, we obtain a contradiction. Thus, Q satisfies Condition C$(0, 0, 0)$. Also, arguing by contradiction, we assume that there exist $\upsilon > 0$ and sequences $\{t_n\}$ and $\{u_n\}$ in $(0, \upsilon)$ satisfying $(t_n, u_n) \in Q$ and $\lim_n (t_n, u_n) = (\upsilon, \upsilon)$. Then we have

$$\tau + \lim_{t \to \upsilon-0} F(t) = \tau + \lim_{n\to\infty} F(u_n) \leq \lim_{n\to\infty} F(t_n) = \lim_{t\to\upsilon-0} F(t),$$

which implies a contradiction. Therefore we have shown that Q satisfies Condition C$(1, 0, 0)$.

In order to prove (ii), we assume that F is right continuous. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and a sequence $\{t_n\}$ in (υ, ∞) satisfying $(t_n, \upsilon) \in Q$ and $\lim_n t_n = \upsilon$. Then we have

$$\tau + F(\upsilon) \leq \lim_{n\to\infty} F(t_n) = \lim_{t\to\upsilon+0} F(t) = F(\upsilon),$$

which implies a contradiction. Therefore we obtain (ii).

In order to prove (iii), we assume that F is left continuous. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and a sequence $\{u_n\}$ in $(0, \upsilon)$ satisfying $(\upsilon, u_n) \in Q$ and $\lim_n u_n = \upsilon$. Then we have

$$\tau + F(\upsilon) = \tau + \lim_{t\to\upsilon-0} F(t) = \tau + \lim_{n\to\infty} F(u_n) \leq F(\upsilon),$$

which implies a contradiction. Therefore we obtain (iii).

In order to prove (iv), we assume that F is continuous. Arguing by contradiction, we assume that there exist $\upsilon > 0$ and sequences $\{t_n\}$ and $\{u_n\}$ in $(0, \infty)$ satisfying $(t_n, u_n) \in Q$ and $\lim_n (t_n, u_n) = (\upsilon, \upsilon)$. Then we have

$$\tau + F(\upsilon) = \tau + \lim_{n\to\infty} F(u_n) \leq \lim_{n\to\infty} F(t_n) = F(\upsilon),$$

which implies a contradiction. Therefore we obtain (iv). □

4. Proof of Theorem 1.2

In this section, in order to clarify the mathematical structure of Theorem 1.2, we give a proof of Theorem 1.2.

Theorem 4.1 (Theorem 2 in Jachymski [4])**.** *Let (X, d) be a complete metric space and let T be a continuous mapping on X. Define L by* (1.1). *Assume the following*:

(i) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < L(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$.*
(ii) *$L(x, y) > 0$ implies $d(Tx, Ty) < L(x, y)$.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

Now we give a proof of Theorem 1.2.

Proof of Theorem 1.2. Define a subset Q of $[0, \infty)^2$ by (2.1).

We first assume (F6). Then by Lemma 3.7, Q satisfies Condition C$(1, 0, 0)$. So Q satisfies Condition C$(0, 0, 0)$. By Proposition 3.6, Q is CJM. Thus, all the assumption of Theorem 4.1 holds. By Theorem 4.1, we obtain the desired result.

We next assume (F7). Then by Lemma 3.7, Q satisfies Condition C$(1, 1, 2)$. In particular, Q satisfies Condition C$(0, 1, 0)$. By Proposition 3.6, Q is of New-type. Using Theorem 2.1, we obtain the desired result. □

Remark 4.2. *We do not need* (F3).

5. Example

The following example tells that the assumption of Theorem 2.1 is reasonably weak. Also the example implies that we cannot generalize Theorem 1.1 with using L.

Example 5.1. *Put $X = [0, 1]$ and let d be as usual. Define a mapping T on X by*

$$Tx = \begin{cases} 1 & \text{if } x = 0 \\ x/2 & \text{if } x > 0. \end{cases}$$

Define L and Q by (1.1) *and* (2.1)*, respectively. Then the following assertions hold*:

(i) *(X, d) is a complete metric space.*
(ii) *T does not have a fixed point.*
(iii) *Q does not satisfy Condition $C(0, 1, 0)$.*
(iv) *Q satisfies Condition $C(1, 0, 2)$.*

Proof. (i) and (ii) are obvious. Let us prove (iii) and (iv). For $x \in X \setminus \{0\}$, we have

$$L(x, 0) = d(0, T0) = 1,$$
$$d(Tx, T0) = d(x/2, 1) = 1 - x/2.$$

Define a sequence $\{(t_n, u_n)\}$ in Q by

$$t_n := L(2^{-n}, 0) = 1,$$
$$u_n := d(T2^{-n}, T0) = 1 - 2^{-n-1}.$$

Then, since $\{u_n\}$ converges to 1, Q does not satisfy Condition C$(0, 1, 0)$. For $x, y \in X \setminus \{0\}$, we have

$$d(Tx, Ty) = (1/2)\, d(x, y) \le (1/2)\, L(x, y).$$

So Q satisfies Condition C$(1, 0, 2)$. □

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