



Survey: On some Midpoint-type algorithms

Giuseppe Marino^{a,b}, Roberta Zaccone^a

^aUniversità della Calabria, Dipartimento di Matematica e Informatica, Arcavacata di Rende (CS), 87036, Italy

^bKing Abdulaziz University, Department of Mathematics, P.O. Box 80203, Jeddah 21589, Saudi Arabia.

Abstract

We introduce iterative methods approximating fixed points for nonlinear operators defined on infinite-dimensional spaces. The starting points are the Implicit and Explicit Midpoint Rules, which generate polygonal functions approximating a solution for an ordinary differential equation in finite-dimensional spaces. The purpose is to determine suitable conditions on the mapping and the underlying space, in order to get strong convergence of the generated sequence to a common solution of a fixed point problem and a variational inequality. The authors contributions appear in the papers [34], [60], [61].

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1. Introduction

Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space, $C \subset X$ a nonempty and closed set, $T : C \rightarrow C$ a nonlinear operator with $Fix(T) = \{z \in C : Tz = z\} \neq \emptyset$.

A classical problem in Metric Fixed Point Theory can be formulated as:

Examine the conditions under which the equation $x = Tx$ may be solved by successive approximations:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = Tx_n, \quad n \geq 0. \end{cases} \quad (1.1)$$

Recall that a mapping $T : C \rightarrow C$ is said L-Lipschitzian if there exists a constant $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in E.$$

In particular,

Email addresses: giuseppe.marino@unical.it (Giuseppe Marino), zaccone@mat.unical.it (Roberta Zaccone)

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if $L < 1$ then T is called contraction;

if $L = 1$ then T is said nonexpansive.

The first result, dealing with the convergence of sequence (1.1), is the well known Banach Principle ([5]). It holds for contractions defined on a complete metric space. Nevertheless, under the same hypothesis, if the mapping T is nonexpansive, then it is not guaranteed neither the existence nor the uniqueness of a fixed point; moreover, the sequence of iterates (1.1) may fail to converge to the fixed point even if it exists (see [38, example 2.1] and [18, example 6.4])

Recognition of fixed point existence results for nonexpansive mappings has a significant line of research in the works of Browder [8], Gohde [37] and Kirk [51] published in 1965. We recall the following results, which were proved independently:

Theorem 1.1 (Browder-Gohde's theorem). *If C is a bounded, closed and convex subset of a uniformly convex Banach space X and if $T : C \rightarrow C$ is nonexpansive, then T has a fixed point.*

Theorem 1.2 (Kirk's theorem). *Let C be a weakly-compact, convex subset of a Banach space X . Assume that C has the normal structure property, then any nonexpansive mapping $T : C \rightarrow C$ has a fixed point.*

Nonexpansive mappings, besides being a generalization of contractions, represent a class of interest for its connection with

- Evolution inclusions: $0 \in \frac{du}{dt} + T(t)u$, where $T(t)$ is, in general, set-valued and accretive or dissipative and minimally continuous.
- Convex minimization problems: let C a closed and convex subset of a real Hilbert space H , $\phi : C \rightarrow \mathbb{R}$ a convex and Fréchet differentiable function, finding $x_0 \in C$ such that

$$\phi(x_0) = \min_{x \in C} \phi(x),$$

that is equivalent to solve

$$\langle \nabla \phi(x_0), y - x_0 \rangle \geq 0, \quad \forall y \in C,$$

can be treated as the fixed point problem

$$x_0 = P_C(x_0 - \frac{1}{L_F} \nabla \phi(x_0)),$$

where $\frac{1}{L_F}$ is the Lipschitz constant for $\nabla \phi$.

These facts promoted the development of two basic research directions:

- Study of suitable assumptions regarding the structure of the underlying space X and/or restrictions on T to ensure the existence of at least a fixed point;
- Construction of iterative methods for approximating the fixed points of T .

Historically, one of the most investigated methods approximating fixed points of nonexpansive mappings dates back to 1953 and is known as Mann's method, in light of Mann [58]. Let C be a nonempty, closed and convex subset of a Banach space X , Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T(x_n), \quad n \geq 0, \end{cases} \quad (1.2)$$

where $(\alpha_n)_{n \in \mathbb{N}}$ is a real control sequence in $(0, 1)$.

Given a nonexpansive mapping T with at least a fixed point, from [72] it is known that if $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = +\infty$ and C is a subset of a uniformly convex Banach space with Fréchet differentiable norm, then the sequence generated by (1.2) weakly converges to a fixed point of T . This result has been generalized in 2001 [30, Theorem 6.4], under the same hypotheses of the parameters sequence, to a more general setting of Banach spaces.

However the convergence is not strong, in general, even in a Hilbert space setting, as shows the celebrated counterexample in [35]. Since then, many modifications to the original Mann’s algorithm have been provided in order to get strong convergence (see the books [1], [6] and the papers [16], [21], [40], [48], [50], [62], [65], [66], [81], [92], [93] with references therein). In detail, we mention the schemes obtained by:

Ishikawa ([46]):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(\beta_n x_n + (1 - \beta_n)Tx_n), \quad n \geq 0. \end{cases}$$

Halpern, ([39]):

$$\begin{cases} x_0, u \in C, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n \quad n \geq 0. \end{cases}$$

Moudafi, ([64]):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n \quad n \geq 0, \end{cases}$$

where $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}} \in (0, 1)$ and $f : C \rightarrow C$ is a contraction.

More recently there exist other attempts to give iterative methods for nonexpansive mappings arising from a different perspective. In detail, consider an initial value problem for ODE’s (Ordinary Differential Equations) of the type

$$\begin{cases} x'(t) = \Phi(x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1.3}$$

Most of equations of type (1.3) cannot be solved in closed form, therefore numerical integration becomes an important tool in order to get informations about the solution trajectory. To this regard, we recall the Midpoint Numerical Rules. Given a time interval $[t_0, T]$, these procedures compute for each positive integer N :

- The step-size $h = \frac{T-t_0}{N}$,
- The time nodes $\{t_n = t_0 + nh\}_{n=0}^N$,
- Approximate values $\{y_n\}_{n=0}^N$ of the solution $x(t)$, $y_n \approx x(t_n)$;
- The polygonal $Y_N(t)$, connecting each pair of consecutive points $(t_n, y_n), (t_{n+1}, y_{n+1})$, for $n = 0, 1, \dots, N$.

A midpoint numerical rule differs from another one for the way in which approximate values $\{y_n\}_{n=0}^N$ of the solution are given. Therefore we count:

Implicit Midpoint Rule (IMR):

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n + h\Phi\left(\frac{y_n + y_{n+1}}{2}\right), \quad n = 0, \dots, N - 1. \end{cases} \tag{1.4}$$

Explicit Midpoint Rule (EMR):

$$\begin{cases} y_0 = x_0, \\ \bar{y}_{n+1} = y_n + h\Phi(y_n), \\ y_{n+1} = y_n + h\Phi\left(\frac{y_n + \bar{y}_{n+1}}{2}\right), \quad n = 0, \dots, N-1. \end{cases} \quad (1.5)$$

In both cases, the following theorem holds:

Theorem 1.3. [67] *If Φ is a Lipschitz continuous and sufficiently smooth function, then the sequence $\{Y_N\}_{N \in \mathbb{N}}$ converges to the exact solution of (1.3), as $N \rightarrow \infty$, uniformly on $t \in [t_0, T]$, for any fixed $T > 0$.*

It can be noticed that if $\Phi = I - g$, with I identity operator, then finding the critical points for

$$\begin{cases} x'(t) = \Phi(x(t)) \\ x(t_0) = x_0 \end{cases}$$

is equivalent to solve the fixed point problem for g , $x = g(x)$. This fact motivated M. A. Alghamdi, M. A. Alghamdi, N. Shahzad and H-K Xu, in [3], to introduce a fixed point iteration for nonexpansive mappings starting from formal analogy with the IMR scheme. The proposed method is implicit and is a Mann-type scheme, named Implicit Midpoint Rule for nonexpansive mappings:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \end{cases} \quad (1.6)$$

where $T : H \rightarrow H$ is a nonexpansive mapping and $(t_n)_{n \in \mathbb{N}}$ is a sequence in $(0, 1)$.

For this procedure, Alghamdi et al. proved the following weak convergence result:

Theorem 1.4. *Let H be a Hilbert space and $T : H \rightarrow H$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$.*

Let $(x_n)_{n \in \mathbb{N}}$ be the sequence generated by

$$\begin{cases} x_0 \in H \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \end{cases}$$

with $(\alpha_n)_{n \in \mathbb{N}} \in (0, 1)$ satisfying the conditions

- $\alpha_{n+1}^2 \leq a\alpha_n$, $\forall n \geq 0$ and some $a > 0$,
- $\liminf_{n \rightarrow \infty} \alpha_n > 0$.

The sequence $(x_n)_{n \in \mathbb{N}}$ weakly converges to a fixed point of T .

Our purpose is to provide a variation to (1.6) in order to get strong convergence. The modification line is analogous to that adopted in 2015 by N. Hussain, G. Marino, L. Muglia, L. Alamri in their work [43]: the proposed algorithm differs from scheme (1.6) for the introduction of a term $\alpha_n \mu_n (u - x_n)$ that can also be infinitesimal. The framework is still that of a Hilbert space H . The obtained scheme is given by

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right) + \alpha_n \mu_n (u - x_n), \quad n \geq 0, \end{cases} \quad (1.7)$$

where $(\alpha_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$ are sequences in $(0, 1]$ and $T : H \rightarrow H$ is a nonexpansive mapping.

We show that, under suitable conditions on the parameters $(\alpha_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$, the sequence $(x_n)_{n \in \mathbb{N}}$, generated by (1.7), converges strongly to the fixed point of T nearest to u .

A different situation occurs if the starting point is the less known numerical EMR. Through formal analogy with scheme (1.5), a recursive procedure for the fixed point problem $x = Tx$ is obtained and it is given by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = (1 - t_n)x_n + t_nT(x_n) \\ x_{n+1} = (1 - t_n)x_n + t_nT\left(\frac{x_n + \bar{x}_{n+1}}{2}\right), \quad n \geq 0, \end{cases} \tag{1.8}$$

with $(t_n)_{n \in \mathbb{N}} \in (0, 1)$ and $T : H \rightarrow H$ is a nonexpansive mapping.

We designate it with Explicit Midpoint Rule for nonexpansive mappings.

Moreover, if the midpoint $\frac{x_n + \bar{x}_{n+1}}{2}$ in the evaluation of T in (1.8) is replaced with any convex combination between x_n and \bar{x}_{n+1} , then scheme (1.8) is named General Explicit Midpoint Rule for nonexpansive mappings.

We provide for the latter scheme the same formal modification as for the IMR for nonexpansive mappings, following [43]. The proposed method is given by

$$\begin{cases} x_0, u \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}) + \alpha_n \mu_n(u - x_n), \quad n \geq 0, \end{cases} \tag{1.9}$$

where $(\alpha_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}$ are sequences in $(0, 1]$.

Even in this case, we show that the sequence generated by (1.9) strongly converges to the fixed point of T closest to u .

Inspired by work [91] of H. K. Xu, M. A. Alghamdi, N. Shahzad and the paper [49] of Y. Ke and C. Ma, we propose another explicit iterative method, starting from the EMR scheme for nonexpansive mappings. It is called Generalized Viscosity Explicit Midpoint Rule (GVEMR) and is given by

$$\begin{cases} x_0 \in C \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, \quad n \geq 0, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}), \quad n \geq 0. \end{cases} \tag{1.10}$$

Iteration (1.10) is obtained, from (1.8), introducing a viscosity term $f \in \Pi_C$ and replacing the midpoint of $[x_n, \bar{x}_{n+1}]$ with a generic point of the same interval in the evaluation of T .

The purpose is to approximate fixed points of quasi-nonexpansive mappings in Hilbert spaces. We recall that a mapping $T : C \rightarrow C$, with C a nonempty subset of a Banach space X , is said to be quasi-nonexpansive if T has at least a fixed point and verify

$$\|Tx - q\| \leq \|x - q\|, \quad \forall q \in \text{Fix}(T), \forall x \in C.$$

This class of mappings, besides for including the class of nonexpansive operators with at least a fixed point, is of interest for the researchers because they can be discontinuous (see, for examples, the pioneering works [25], [27] and more recently the monograph [18]). In literature can be found several works dealing with the fixed points approximation of a quasi-nonexpansive operators (see, for instance, [82], [26], [83], [56], [85]).

In a second stage, strong convergence results are proved for the class of quasi-nonexpansive mappings in the more general setting of p -uniformly convex Banach spaces, for $1 < p < \infty$, with new techniques with respect to those employed in a Hilbert spaces framework.

The first main result is applicable to l_p spaces; the second one, using the concept of ψ -expansive mappings (see [33] and references therein), is applicable to L_p spaces which fail to have a weakly continuous duality mapping.

2. Preliminaries

Throughtout the next sections, will be denoted with

- H , a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$;
- X , a real Banach space with norm $\| \cdot \|$;
- X^* , the dual space of X with duality pairing $\langle x, x^* \rangle = x^*(x)$ for each $x \in X$ and $x^* \in X^*$;
- C , a closed and convex subset of H or X ;
- $T : C \rightarrow C$, a nonexpansive or a quasi-nonexpansive mapping;
- $f : C \rightarrow C$, a θ -contraction for a certain $\theta \in [0, 1]$;
- $Fix(T)$, the fixed points set of T ;
- $P_{Fix(T)}$, the metric projection of C onto $Fix(T)$;
- $Q : C \rightarrow Fix(T)$, a sunny nonexpansive retraction;
- J_ϕ , the duality mapping associated to the gauge function ϕ ;
- J , the normalized duality mapping;
- \rightarrow , the strong convergence;
- \rightharpoonup , the weak convergence.

First of all we recall that, in a Hilbert space H , for each $x, y \in H$ and $\lambda \in [0, 1]$, the following inequalities hold:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle \quad (2.1)$$

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \quad (2.2)$$

The framework of the convergence results in Theorem 3.17 and Theorem 3.22 is constituted by p -uniformly convex Banach spaces. About the matter, we need to recall the following definitions:

Definition 2.1. [38] A normed space X is called uniformly convex if for any $\epsilon \in (0, 2]$ there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 2.2. [38] The modulus of convexity of a Banach space X is the function $\delta_X : (0, 2] \rightarrow (0, 1]$ defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}.$$

We mention a characterization for uniformly convex Banach spaces using the notion of modulus of convexity:

Theorem 2.3. [18] A normed space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for each $\epsilon \in (0, 2]$.

Definition 2.4. [18]

Let $p > 1$ be a real number. Then X is said to be p -uniformly convex if there is a constant $c > 0$ such that

$$\delta_X(\epsilon) \geq c\epsilon^p.$$

From the definition, it follows that a p -uniformly convex Banach space is uniformly convex.

Example 2.5. [18]

If $X = L_p$ (or l_p), $1 < p < \infty$, then

1. $\delta_X(\epsilon) \geq \frac{1}{2^{p+1}}\epsilon^2$, if $1 < p < 2$,

2. $\delta_X(\epsilon) \geq \epsilon^p$, if $2 \leq p < \infty$.

In particular, for such class of Banach spaces, we mention that for all $x, y \in X$ and $\lambda \in [0, 1]$, the following inequality is verified ([86]):

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - \lambda(1 - \lambda)c\|x - y\|^p \tag{2.3}$$

for a certain positive constant c , for all $x, y \in X$ and $0 \leq \lambda \leq 1$.

The concept of duality mapping appeared for first time in the work of Beurling and Livingston ([7]).

Definition 2.6. [18] A continuous and strictly increasing function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ such that

$$\phi(0) = 0$$

$$\lim_{t \rightarrow \infty} \phi(t) = +\infty,$$

is called a gauge function (or weight function).

Definition 2.7. [18] Given a gauge function ϕ , the mapping $J_\phi : X \rightarrow 2^{X^*}$ defined by

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|\phi(\|x\|); \phi(\|x\|) = \|x^*\|\}$$

is called the duality mapping with gauge function ϕ .

If the gauge ϕ is given by $\phi(t) = t^{p-1}$, $1 < p < +\infty$ for all $t \in [0, +\infty)$, then $J_\phi = J_p$ is known as p th generalized duality mapping; in particular, for $p = 2$ $J_2 = J$ is called normalized duality mapping.

When we deal with $j_\phi(x)$ we mean a (single-valued) selection of $J_\phi(x)$.

Lemma 2.8. [22] Let ϕ a gauge function and $\Phi(t) = \int_0^t \phi(s) ds$, then Φ is a convex function.

Definition 2.9. [22] The subdifferential of a proper functional $g : X \rightarrow (-\infty, \infty)$ is a map designed with

$$\partial g : X \rightarrow 2^{X^*}$$

and defined by

$$\partial g(x) = \{x^* \in X^* : g(y) \geq g(x) + \langle y - x, x^* \rangle, \forall y \in X\}$$

The duality mapping J_ϕ , associated to ϕ , can be also described in the following way:

Theorem 2.10. [4] If J_ϕ is the duality mapping associated to a gauge ϕ , then

$$J_\phi x = \partial \Phi(\|x\|) \quad \forall x \in X.$$

Thus a subdifferential inequality holds:

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x + y) \rangle, \quad j_\phi(x + y) \in J_\phi(x + y) \tag{2.4}$$

The following definition is due to Browder:

Definition 2.11. [9]

The duality mapping J_ϕ is said to be (sequentially) weak continuous if it is single-valued and maps weakly convergent sequences in X to weak* convergent sequences in X^* , that is, if $x_n \rightharpoonup x$ in X , then $J_\phi(x_n) \rightharpoonup^* J_\phi(x)$ in X^* .

Example 2.12. [90] For each $1 < p < \infty$, the generalized duality map J_p of l_p is weakly continuous, instead that of L_p fails to be weakly continuous.

Example 2.13. [90] Let H be a real (infinite dimensional) Hilbert space. Then J_p is weakly continuous if and only if $p = 2$.

For the fixed points set of a quasi-nonexpansive mapping the following result holds:

Theorem 2.14. [27, Theorem 1]

If C is a closed, convex subset of a strictly convex normed linear space, and $T : C \rightarrow C$ is quasi-nonexpansive, then $Fix(T) = \{z \in C : Tz = z\}$ is a nonempty, closed and convex set in which T is continuous.

Definition 2.15. [38] A nonempty subset K of $C \subset X$ is said to be a retract of C if there exists a continuous mapping $Q : C \rightarrow K$ with $K = Fix(Q)$. Any such mapping Q is a retraction of C onto K

It is known (see [22]) that a Banach space X is smooth if and only if each duality mapping J_ϕ is single-valued. In such spaces, a characterization for a sunny nonexpansive retraction is given by:

Lemma 2.16. [71, Lemma 2.7]

Let X be a smooth Banach space and let C a nonempty subset of X . Let $Q : X \rightarrow C$ a retraction and let J be the normalized duality map on X . The the following are equivalent:

1. Q is sunny and nonexpansive,
2. $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy) \rangle$ for all $x, y \in X$,
3. $\langle x - Qx, J(y - Qx) \rangle \leq 0$ for all $x \in X$ and $y \in C$.

Hence, there is at most one sunny nonexpansive retraction on C .

Remark 2.17. Previous lemma holds even if the normalized duality map J is replaced with the duality map J_ϕ associated to a gauge function ϕ .

Let us recall the definition of ψ -expansive mapping (see papers [31], [32], [33], [54], [80] and references therein).

Definition 2.18. [33] A mapping $A : D(A) \subset X \rightarrow X$ is said to be ψ -expansive if there exists a function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ such that for every $x, y \in D(A)$, the inequality $\|Ax - Ay\| \geq \psi(\|x - y\|)$ holds, with ψ satisfying

- $\psi(0) = 0$;
- $\psi(r) > 0 \quad \forall r > 0$;
- Either ψ is continuous or it is nondecreasing.

Finally, we recall that

Definition 2.19. [12] Let X be a real Banach space and C a nonempty and closed subset of X . A mapping $T : C \rightarrow C$ is said to be demiclosed (at y), if for any $(x_n)_{n \in \mathbb{N}}$, in C , the conditions $x_n \rightharpoonup x$ and $Tx_n \rightarrow y$ imply $Tx = y$.

For a nonexpansive mapping defined on a uniformly convex Banach space, the following holds:

Lemma 2.20. ([12, Theorem 3]) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space X , and let $T : C \rightarrow X$ be a nonexpansive mapping. Then $I - T$ is demiclosed, that is

$$(x_n)_{n \in \mathbb{N}} \subset C, \quad x_n \rightharpoonup x, \quad (I - T)x_n \rightarrow y \implies (I - T)x = y.$$

3. Convergence Results

Let's start recalling iteration (1.7):

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right) + \alpha_n \mu_n (u - x_n), \quad n \geq 0, \end{cases}$$

We can notice that this method is well defined. Indeed, if $T : H \rightarrow H$ is a nonexpansive mapping, y, z, w are given points in H and $\alpha \in (0, 1)$, then the mapping $\tilde{T} : H \rightarrow H$ defined by

$$\tilde{T}x = \alpha y + (1 - \alpha)T\left(\frac{z + x}{2}\right) + w$$

is a contraction with constant $\frac{1-\alpha}{2}$. Therefore \tilde{T} has a unique fixed point. Hence we prove the following result:

Theorem 3.1. [61, Theorem 3.2] *Let H be a real Hilbert space and $T : H \rightarrow H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Assume that the sequences $(\alpha_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}} \in (0, 1]$ satisfy the conditions*

- (1) $\lim_{n \rightarrow \infty} \alpha_n = 0,$
- (2) $\sum_{n=0}^{\infty} \alpha_n \mu_n = +\infty,$
- (3) $\lim_{n \rightarrow \infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n \mu_n} = 0,$
- (4) $\lim_{n \rightarrow \infty} \frac{|\mu_n - \mu_{n-1}|}{\mu_n} = 0.$

Then the sequence $(x_n)_{n \in \mathbb{N}}$, generated by (??), strongly converges to the point $q_u \in Fix(T)$ nearest to u , that is $\|u - q_u\| = \min_{x \in Fix(T)} \|u - x\|$.

A possible choice of parameters satisfying the hypotheses of Theorem 3.1 is given by

$$\alpha_n = \mu_n = \frac{1}{\sqrt{n}}$$

Remark 3.2. We point out that if $u = 0 \in H$, under the same hypotheses of Theorem 3.1, we get that the sequence $(x_n)_{n \in \mathbb{N}}$ generated by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right) - \alpha_n \mu_n x_n \quad n \geq 0 \end{cases}$$

strongly converges to the point $q \in Fix(T)$ nearest to $0 \in H$, that is, the fixed point of T with minimum norm $\|q\| = \min_{x \in Fix(T)} \|x\|$.

A particular case of Theorem 3.1 is obtained for $\mu_n = 1$. The resulting algorithm is a Halpern-type iteration, for which we claim that:

Corollary 3.3. [61, Corollary 3.4] *Let H be a real Hilbert space and $T : H \rightarrow H$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If the sequence $(\alpha_n)_{n \in \mathbb{N}}, \subset (0, 1]$ satisfies the conditions*

- (1) $\lim_{n \rightarrow \infty} \alpha_n = 0,$

$$(2)' \sum_{n=0}^{\infty} \alpha_n = +\infty,$$

$$(3)' \lim_{n \rightarrow \infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0.$$

Then the sequence $(x_n)_{n \in \mathbb{N}}$ generated by

$$\begin{cases} x_0, u \in H, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0 \end{cases} \tag{3.1}$$

strongly converges to the point $q_u \in \text{Fix}(T)$ nearest to u , that is

$$\|u - q_u\| = \min_{x \in \text{Fix}(T)} \|u - x\|$$

Remark 3.4. Even in this case, it is considered the eventuality $u = 0 \in H$. Therefore, we get that, under the same assumptions of Corollary 3.3, the sequence $(x_n)_{n \in \mathbb{N}}$ generated by

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0 \end{cases}$$

strongly converges to the point $q \in \text{Fix}(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|q\| = \min_{x \in \text{Fix}(T)} \|x\|$.

For the sequence generated by (1.9)

$$\begin{cases} x_0, u \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, \quad n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}) + \alpha_n \mu_n(u - x_n), \quad n \geq 0, \end{cases}$$

we prove the following:

Theorem 3.5. [61, Theorem 4.2] Let H be a real Hilbert space and $T : H \rightarrow H$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Under the assumptions (1), (2), (3), (4) of Theorem 3.1, if the sequences $(\alpha_n)_{n \in \mathbb{N}}, (\mu_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}$ $(0, 1]$ satisfy also the hypotheses

$$(5) \lim_{n \rightarrow \infty} \frac{|s_n - s_{n-1}|}{\alpha_n \mu_n} = 0$$

$$(6) \lim_{n \rightarrow \infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n \mu_n} = 0$$

$$(7) \limsup_{n \rightarrow \infty} \beta_n(1 - s_n) + s_n > 0,$$

then $(x_n)_{n \in \mathbb{N}}$ generated by (1.9) strongly converges to the point $x_u^* \in \text{Fix}(T)$ nearest to u , that is

$$\|u - x_u^*\| = \min_{x \in \text{Fix}(T)} \|u - x\|$$

An example of control sequences satisfying conditions (1) – (7) is given by

$$\alpha_n = s_n = \mu_n = \frac{1}{\sqrt{n}}, \quad \beta_n = \frac{n}{n + 1}$$

Remark 3.6. In case $u = 0$, under the same assumptions of Theorem 3.5, we obtain strong convergence of the sequence $(x_n)_{n \in \mathbb{N}}$, generated by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, & n \geq 0 \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}) - \alpha_n \mu_n x_n, & n \geq 0, \end{cases}$$

to the point $x^* \in \text{Fix}(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|x^*\| = \min_{x \in \text{Fix}(T)} \|x\|$.

As in the previous case, in the eventuality $\mu_n = 1$, for all $n \in \mathbb{N}$, we have the following convergence result for a Halpern-type method:

Corollary 3.7. [61, Corollary 4.4] Let H be a real Hilbert space and $T : H \rightarrow H$ a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Assume that conditions (1), (2)', (3)' of Corollary 3.3 hold and that the sequences $(\alpha_n)_{n \in \mathbb{N}}, (\beta_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}$ satisfy also the hypotheses:

$$(5)' \quad \lim_{n \rightarrow \infty} \frac{|s_n - s_{n-1}|}{\alpha_n} = 0,$$

$$(6)' \quad \lim_{n \rightarrow \infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0,$$

$$(7) \quad \limsup_{n \rightarrow \infty} \beta_n(1 - s_n) + s_n > 0,$$

then the sequence $(x_n)_{n \in \mathbb{N}}$ generated by

$$\begin{cases} x_0, u \in H \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, & n \geq 0 \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}), & n \geq 0, \end{cases} \tag{3.2}$$

strongly converges to the point $x_u^* \in \text{Fix}(T)$ nearest to u .

Remark 3.8. For $u = 0$, under the same assumptions of Corollary 3.7, we obtain strong convergence of the sequence $(x_n)_{n \in \mathbb{N}}$, generated by

$$\begin{cases} x_0 \in H, \\ \bar{x}_{n+1} = \beta_n x_n + (1 - \beta_n)Tx_n, & n \geq 0, \\ x_{n+1} = (1 - \alpha_n)T(s_n x_n + (1 - s_n)\bar{x}_{n+1}), & n \geq 0, \end{cases}$$

to the point $x^* \in \text{Fix}(T)$ nearest to $0 \in H$, that is the fixed point of T with minimum norm $\|x^*\| = \min_{x \in \text{Fix}(T)} \|x\|$.

Let us consider the conditions:

$$(i) \quad \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$(ii) \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$$

$$(iii) \quad \limsup_{n \rightarrow \infty} \beta_n(1 - \beta_n)(1 - s_n) > 0.$$

A strong convergence result of the sequence $(x_n)_{n \in \mathbb{N}}$ generated by (1.10) to a fixed point of a quasi-nonexpansive operator is proved in the framework of Hilbert spaces:

Theorem 3.9. [60, Theorem 3.2] Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H . Let $T : C \rightarrow C$ be a quasi-nonexpansive mapping, with $I - T$ demiclosed at 0, and $f : C \rightarrow C$ be a contraction with coefficient $\theta \in [0, 1)$. Let $(\alpha_n)_{n \in \mathbb{N}}$, $(s_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$ be three sequences in $(0, 1)$, satisfying the conditions (i), (ii), (iii). Then, the sequence $(x_n)_{n \in \mathbb{N}}$, defined in (??), strongly converges to $\bar{q} \in \text{Fix}(T)$, which is the unique solution in $\text{Fix}(T)$ of the variational inequality (VI)

$$\langle \bar{q} - f(\bar{q}), \bar{q} - x \rangle \leq 0, \quad \forall x \in \text{Fix}(T). \quad (3.3)$$

Remark 3.10. Conditions

- T is quasi-nonexpansive,
- $I - T$ is demiclosed at 0,

appearing in the previous theorem, are not related. One could consider the following examples to confirm this:

Example 3.11. [41, Example 2.3]

Let $H = \mathbb{R}$, $C = [0, +\infty)$ and $T : C \rightarrow C$ a mapping defined by

$$Tx = \begin{cases} \frac{2x}{x^2+1} & x \in (1, +\infty) \\ 0 & x \in [0, 1] \end{cases}$$

It results that

- $\text{Fix}(T) = \{0\}$;
- T is discontinuous;
- T is quasi-nonexpansive, indeed if $x \in [0, 1]$ then $|Tx - 0| = 0 \leq |x - 0|$; while, if $x \in (1, +\infty)$ then $|Tx - 0| = \frac{2x}{1+x^2} \leq 1 \leq |x - 0|$;
- Considering the sequence $x_n = 1 + \frac{1}{n} \in (1, 2)$, it results that $x_n \rightarrow 1 \notin \text{Fix}(T)$ and $|x_n - Tx_n| \rightarrow 0$, thus $I - T$ is not demiclosed at 0.

Example 3.12. [18, Example 8.2]

Let $H = \mathbb{R}$, $C = [0, 1]$ and $T : C \rightarrow H$ defined by $Tx = 1 - x^{\frac{2}{3}}$.

- $\text{Fix}(T) = \{q\}$, with $q \in (0, 1)$;
- T is a continuous pseudo-contraction, since $I - T$ is monotone; therefore $I - T$ is demiclosed at 0 (see [93, Demi-closedness Principle]);
- T is not quasi-nonexpansive since:
 - if $x = 0$, then $|Tx - q| \leq |x - q|$ implies $1 - q \leq q$, and hence $q \geq \frac{1}{2}$
 - if $x = 1$, then $|Tx - q| \leq |x - q|$ implies $q \leq 1 - q$, and hence $q \leq \frac{1}{2}$

Thus it must be $q = \frac{1}{2}$, that is a contradiction.

Remark 3.13. There exist mappings $T : C \rightarrow C$, with $\text{Fix}(T)$ nonempty, which are quasi-nonexpansive and such that $I - T$ is demiclosed at 0. Among these, in addition to nonexpansive mappings, let us mention

- Nonspreading mappings, introduced by Kohsaka and Takahashi in 2008:

Definition 3.14. [52] Let X be a smooth, strictly convex and reflexive Banach space, let J be the duality mapping of X and let C a nonempty, closed and convex subset of X . Then, a mapping $S : C \rightarrow C$ is said to be nonspreading if

$$\Phi(Sx, Sy) + \Phi(Sy, Sx) \leq \Phi(Sx, y) + \Phi(Sy, x),$$

for all $x, y \in C$, where $\Phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in X$.

Particularly, if X is a Hilbert space, it is known that $\Phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Then a nonspreading mapping $S : C \rightarrow C$ in a Hilbert space is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2,$$

for all $x, y \in C$.

- L-hybrid mappings, introduced by Aoyam et al. in 2010:

Definition 3.15. [2] Let $T : H \rightarrow H$ be a mapping and L a nonnegative number. We will say that T is L-hybrid, signified as $T \in \mathcal{H}_L$, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + L\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in H.$$

Theorem 3.9 holds for these classes of mappings.

Remark 3.16. In Theorem 3.9, no additional assumption has been formulated for the limit of the sequence $(s_n)_{n \in \mathbb{N}}$, hence it could be that $\lim_{n \rightarrow \infty} s_n = 0$.

Theorem 3.9 has been improved by Garcia-Falset, Marino and Zaccone in [34] in the more general setting of p -uniformly convex Banach spaces, in which inequalities analogous to (2.1) and (2.2), valid in Hilbert spaces, hold (see Xu's paper [86] for a detailed survey).

Theorem 3.17. [34, Theorem 3.2] Let X be a p -uniformly convex Banach space, with $1 < p < +\infty$, having a weakly sequentially continuous duality mapping J_ϕ . Let C be a nonempty, closed and convex subset of X , $T : C \rightarrow C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0,1 and $f : C \rightarrow C$ a θ -contraction, for a certain $\theta \in [0, 1)$.

Let $(x_n)_{n \in \mathbb{N}}$ be generated by (1.10). Assume that the sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$, $(s_n)_{n \in \mathbb{N}}$ satisfy the conditions (i), (ii), (iii). If $\text{Fix}(T)$ is the sunny nonexpansive retract of C , with $Q : C \rightarrow \text{Fix}(T)$ sunny nonexpansive retraction, then $(x_n)_{n \in \mathbb{N}}$ strongly converges to $\bar{q} = Q(f(\bar{q}))$. Further \bar{q} is the unique solution in $\text{Fix}(T)$ of the variational inequality

$$\langle f(\bar{q}) - \bar{q}, j_\phi(x - \bar{q}) \rangle \leq 0, \quad \forall x \in \text{Fix}(T) \quad (3.4)$$

Remark 3.18. In order to have a strong convergence result in the setting of a p -uniformly Banach space X , we assume that X has weakly sequentially continuous duality mapping J_ϕ , for a certain gauge function ϕ . This hypothesis, compared to the weak continuity for the normalized duality map J , that is frequently assumed in trying to extend some results from the setting of Hilbert spaces to that of Banach spaces (see [47], [78] and other works), allows us to include, for instance, also the sequential spaces l_p . Indeed l_p spaces, for $p \neq 2$, fail to have weakly continuous map J , but they have generalized duality map J_p weakly sequentially continuous (for a detailed survey, see [90]).

Concerning the hypotheses assumed for the control sequence in Theorem (3.17), as well as in Theorem 3.9, no additional assumption has been formulated for the limit of sequence $(s_n)_{n \in \mathbb{N}} \subset (0, 1)$, that, for instance, may converge to zero. (see Example 4.2).

In light of Theorem (3.17), we include the following corollary:

Corollary 3.19. [34, Corollary 3.4] Let X be a p -uniformly convex Banach space, for $1 < p < \infty$, having a weakly sequentially continuous duality mapping J_ϕ . Let C be a nonempty, closed and convex subset of X , $T : C \rightarrow C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0 and $\text{Fix}(T) = \{q\}$. Let $f : C \rightarrow C$ a θ -contraction, for a certain $\theta \in [0, 1)$.

Let $(x_n)_{n \in \mathbb{N}}$ be the sequence generated by (1.10).

If the sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$, $(s_n)_{n \in \mathbb{N}}$ satisfy the conditions (i), (ii), (iii), then $(x_n)_{n \in \mathbb{N}}$ strongly converges to q .

Remark 3.20. The assert follows from Theorem (3.17), considered that $Fix(T) = \{q\}$ is a sunny nonexpansive retract of C , with sunny nonexpansive retraction the constant function $Q(x) = q$.

If $X = H$ is a real Hilbert space, then X is 2-uniformly convex, with normalized duality mapping weakly sequentially continuous. Let C be a closed and convex subset of H . From [27, Theorem 1], it is known that if $T : C \rightarrow C$ is a quasi-nonexpansive then $Fix(T)$ is nonempty, closed and convex. Therefore the metric projection $P_{Fix(T)}$ is a sunny nonexpansive retraction from C onto $Fix(T)$. Since there is at most one sunny nonexpansive retraction (Lemma 2.16), we have that $Q \equiv P_{Fix(T)}$.

These considerations motivate the result that follows.

Corollary 3.21. [34, Corollary 3.5] *Let H be real Hilbert space. Let C be a nonempty, closed and convex subset of H , $T : C \rightarrow C$ a quasi-nonexpansive mapping, such that $I - T$ is demiclosed at 0, and $f : C \rightarrow C$ a θ -contraction, for a certain $\theta \in [0, 1)$.*

Let $(x_n)_{n \in \mathbb{N}}$ be generated by 1.10. Assume that the sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$, $(s_n)_{n \in \mathbb{N}}$ satisfy the conditions (i), (ii) and (iii). Then $(x_n)_{n \in \mathbb{N}}$ strongly converges to $q \in Fix(T)$. Further q is the unique solution in $Fix(T)$ of the variational inequality

$$\langle f(q) - q, x - q \rangle \leq 0, \quad \forall x \in Fix(T).$$

Therefore Theorem 3.9 can be considered as a particular case of Theorem 3.17.

For the eventuality in which the Banach space X fails to have weakly sequentially continuous duality map J_ϕ , as occurs for L_p spaces, we establish a strong convergence result for the GVEMR, assuming the additional assumption that $I - T$ is ψ -expansive (for more details on this type of mapping, see [33] and references therein).

Theorem 3.22. [34, Theorem 3.7] *Let X be a p -uniformly convex Banach space, for $1 < p < \infty$, and $C \subset X$ a nonempty, closed and convex set. Let $T : C \rightarrow C$ a quasi-nonexpansive mapping such that $I - T$ is ψ -expansive, and $f : C \rightarrow C$ a θ -contraction for a certain $\theta \in (0, 1)$.*

Let $(x_n)_{n \in \mathbb{N}}$ the sequence generated by (1.10). If the parameters sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ satisfy the conditions (i), (iii), then $(x_n)_{n \in \mathbb{N}}$ strongly converges to to the unique fixed point of T .

4. Examples

Inspired to [44, Example 2] by Iemoto and Takahashi, we give an example for the convergence result stated in Theorem 3.9 for the class of nonspreading operators:

Example 4.1. Let H be a Hilbert space. Assume that

$$B_1 = \{x \in H \quad s.t. \quad \|x\| \leq 1\},$$

$$B_2 = \{x \in H \quad s.t. \quad \|x\| \leq 2\},$$

$$B_3 = \{x \in H \quad s.t. \quad \|x\| \leq 3\}.$$

The mapping defined as

$$Sx = \begin{cases} 0 & \text{if } x \in B_2 \\ P_{B_1}(x) & \text{if } x \in B_3 \setminus B_2, \end{cases}$$

with P_{B_1} the metric projection of H onto B_1 , is a nonspreading operator with $Fix(S) = \{0\}$, hence it is quasi-nonexpansive. It is known that $I - S$ is demiclosed at 0.

Moreover S is discontinuous, hence it is not nonexpansive.

Let us choose $H = \mathbb{R}$, $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n-1}{2n}$, $s_n = \frac{1}{n}$, $f(x) = \frac{x}{2}$, $x_0 = 3$.

The operator S can be written as

$$Sx = \begin{cases} 0 & \text{if } x \in [-2, 2] \\ P_{B_1}(x) & \text{if } x \in [-3, 3] \setminus [-2, 2], \end{cases}$$

with $B_1 = [-1, 1]$, $B_2 = [-2, 2]$, $C = [-3, 3]$.

While the sequence generated by (1.10) is given by

$$\bar{x}_{n+1} = \frac{n}{2(n+1)}x_n + \frac{n+2}{2(n+1)}Sx_n, \quad n \geq 0$$

$$x_{n+1} = \frac{x_n}{2(n+1)} + \left(\frac{n}{n+1}\right)S\left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right), \quad n \geq 0.$$

Therefore the sequence $(x_n)_{n \in \mathbb{N}}$ is given by

$$\begin{aligned} x_0 &= 3, \\ x_1 &= \frac{3}{2}, \\ x_2 &= \frac{3}{8}, \\ &\dots \\ x_{n+1} &= \frac{3}{(n+1)!2^{n+1}} \end{aligned}$$

that quickly converges to 0.

For Theorem (3.17) we include the following:

Example 4.2. Consider the real Banach space $X = l_p$, for $1 < p < +\infty$ endowed with the norm $\|x\| = \|x\|_p = [\sum_{i=1}^{\infty} |x_i|^p]^{\frac{1}{p}}$, for $x = (x_1, x_2, \dots, x_n, \dots)$.

Set

$$B_1 = \{x \in l_p : \|x\| \leq 1\}$$

$$B_2 = \{x \in l_p : \|x\| \leq 2\}$$

$$B_3 = \{x \in l_p : \|x\| \leq 3\}$$

Let $T : B_3 \rightarrow B_3$ the mapping defined as

$$Tx = \begin{cases} 0_{l_p} & x \in B_2 \\ P_{B_1}(x) & x \in B_3 \setminus B_2 \end{cases}$$

where $P_{B_1}(x)$ is the metric projection of X onto B_1 .

Let $f : B_3 \rightarrow B_3$ the map defined as $f(x) = \frac{1}{2}x$.

Let us put $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2(n+1)}$, $s_n = \frac{1}{n+1}$.

We point out the following considerations:

- l_p , with $1 < p < \infty$, is a p -uniformly convex Banach space with weakly continuous duality mapping J_p ,
- $Fix(T) = \{0\}$,
- T is quasi-nonexpansive,
- T is not nonexpansive since it is discontinuous,

- $I - T$ is demiclosed at 0.

Indeed if we consider $x_n \rightharpoonup x \in B_3 \setminus B_2$ then we have that

$$\|x_n - Tx_n\| \geq \|x_n\| - \|Tx_n\| \geq \|x_n\| - 1,$$

so that $\liminf_{n \rightarrow +\infty} \|x_n - Tx_n\| \geq \|x\| - 1 \geq 1$.

We approach to the same conclusion if $x_n \rightharpoonup x \in B_2$, with $x \neq 0$, considering that $\liminf_{n \rightarrow +\infty} \|x_n - Tx_n\| \geq \|x\| > 0$.

- f is a $\frac{1}{2}$ -contraction,
- sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$, $(s_n)_{n \in \mathbb{N}}$ satisfy conditions (i), (ii) and (iii) of Theorem 3.17.

Fixed $x_0 = (3, 0, 0, \dots)$, then algorithm (1.10) is given by

$$\begin{cases} \bar{x}_{n+1} = \frac{n}{2(n+1)}x_n + \frac{n+2}{2(n+1)}Tx_n, & n \geq 0 \\ x_{n+1} = \frac{x_n}{2(n+1)} + \frac{n}{n+1}T\left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right) & n \geq 0 \end{cases}$$

It generates the sequence:

$$\begin{aligned} x_0 &= (3, 0, 0, \dots), \\ x_1 &= \frac{1}{2}(3, 0, 0, \dots), \\ x_2 &= \frac{1}{8}(3, 0, 0, \dots), \\ &\dots \\ x_{n+1} &= \frac{1}{(n+1)!2^{n+1}}(3, 0, 0, \dots), \end{aligned}$$

that converges to 0, as $n \rightarrow +\infty$.

For Theorem (3.22), we give the following example in a p -uniformly convex Banach space that fails to have a weakly continuous duality mapping:

Example 4.3. Let $X = L_p([0, 1])$, with $1 < p < +\infty$, endowed with the norm

$$\|x\| = \|x\|_p = \left[\int_{[0,1]} |x(s)|^p ds \right]^{\frac{1}{p}}$$

and

$$\begin{aligned} B_1 &= \{x \in L_p[0, 1] : \|x\| \leq 1\} \\ B_2 &= \{x \in L_p[0, 1] : \|x\| \leq 2\} \end{aligned}$$

Let $T : B_2 \rightarrow B_2$ the map defined as

$$Tx = \begin{cases} 0 & x \in B_1 \\ -x & x \in B_2 \setminus B_1 \end{cases} \tag{4.1}$$

and $f : B_2 \rightarrow B_2$ such that $f(x) = \frac{x}{2}$.

If we set $\alpha_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{n}{2(n+1)}$, $s_n = \frac{1}{n+1}$ then algorithm (1.10) becomes

$$\begin{cases} \bar{x}_{n+1} = \frac{n}{2(n+1)}x_n + \frac{n+2}{2(n+1)}Tx_n, & n \geq 0 \\ x_{n+1} = \frac{x_n}{2(n+1)^2} + \left(\frac{n^2+2n}{(n+1)^2}\right)T\left(\frac{x_n}{n+1} + \frac{n}{n+1}\bar{x}_{n+1}\right) & n \geq 0 \end{cases}$$

We observe that

- L_p is a p -uniformly convex Banach space that fails to have a weakly continuous duality mapping,
- $Fix(T) = \{0\}$,
- T is quasi-nonexpansive,
- T is discontinuous so it is not nonexpansive,
- $I - T$ is ψ -expansive. Indeed:
 - if $x, y \in B_1$, then $\|(I - T)x - (I - T)y\| = \|x - y\|$,
 - if $x, y \in B_2$, then $\|(I - T)x - (I - T)y\| = 2\|x - y\|$,
 - if $x \in B_1$ and $y \in B_2$, then

$$\begin{aligned} \|(I - T)x - (I - T)y\| &= \|x - 2y\| \\ &\geq 2\|y\| - \|x\| \\ &\geq 1 \end{aligned}$$

Since

$$\|x - y\| \leq 3 \quad \forall x \in B_1, y \in B_2,$$

then $\|(I - T)x - (I - T)y\| \geq \frac{1}{3}\|x - y\|$.

Therefore $I - T$ is ψ -expansive with $\psi(t) = \frac{t}{3}$, for $t \in [0, +\infty)$,

- f is a $\frac{1}{2}$ -contraction,
- The sequences α_n , β_n and s_n satisfy conditions 1) and 2) of the preceding theorem.

Fixed the constant function $x_0 = 2$ in $L_p([0, 1])$, then the other terms of the sequence $(x_n)_{n \in \mathbb{N}}$ are given by

$$\begin{aligned} x_0 &= 2 \\ x_1 &= 1 \\ x_2 &= \frac{1}{8} \\ x_3 &= \frac{1}{144} \\ &\dots \end{aligned}$$

It can be noticed that, for $n \geq n_0 \in \mathbb{N}$, the term $T\left(\frac{x_n}{n} + \frac{n-1}{n}\bar{x}_{n+1}\right)$ vanishes since $\left\|\frac{x_n}{n} + \frac{n-1}{n}\bar{x}_{n+1}\right\| \leq 1$, then $x_{n+1} = \frac{x_n}{2n^2}$ and $\lim_{n \rightarrow +\infty} \left\|\frac{x_n}{2n^2}\right\| = 0$.

References

- [1] R.P. Agarwal, D. O'Regan, D.R. Sahu, Fixed Point Theory for Lipschitzian-type Mappings with Applications, Series Topological Fixed Point Theory and Its Applications, Springer, New York, 2009.
- [2] K. Aoyama, S. Iemoto, F. Kohsaka, W. Takahashi, Fixed point and ergodic theorems for λ -hybrid mappings in Hilbert spaces, J. Nonlinear Convex Anal., 11 (2010) 335-343.
- [3] M.A. Alghamdi, M.A. Alghamdi, N. Shahzad, H.K. Xu, The implicit midpoint rule for nonexpansive mappings, Fixed Point Theory Appl. 2014, 96 (2014).
- [4] E. Asplund, Positivity of duality mappings, Bull. Amer. Math. Soc, 73 (1967), 200-203.
- [5] S. Banach Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fundamenta Mathematicae, (1922).
- [6] V. Berinde, Iterative Approximation of Fixed Points, second ed., in: Lecture Notes in Mathematics, vol. 1912, Springer, Berlin, 2007.
- [7] A. Beurling and A. E. Livingston, A theorem on duality mappings in Banach spaces, Ark. Mat. 4 (1961), 405-411.
- [8] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. U.S.A., 54 (1965) 1041-1044.

- [9] F.E. Browder, Fixed point theorems for nonlinear semicontractive mappings in Banach spaces, Arch. Rational Mech. Anal. 21 (1966) 259-269.
- [10] Browder, F.E., Convergence of approximants to fixed points of nonexpansive nonlinear maps in Banach spaces, Arch. Rat. Mech. Anal., 24 (1967) 82-90.
- [11] F. E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73 (1967) 875-882.
- [12] F.E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, Bull. Amer. Math. Soc. 74 (1968) 660-665.
- [13] F. E. Browder and W. V. Petryshyn, The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Soc. 72 (1966) 571-575.
- [14] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, J. Math. Anal. Appl. 20 (1967) 197-228.
- [15] R. E. Bruck, Asymptotic behavior of nonexpansive mappings, Contemporary Math. 18 (1983) 1-47.
- [16] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, Inverse Problems, 20 (2004) 103-120.
- [17] B. Carnahan, A. H. Luther, and J. Wilkes, Applied Numerical Methods, John Wiley and Sons, New York, 1969.
- [18] C. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, vol. 1965 of Lecture Notes in Mathematics, Springer, London, UK, 2009.
- [19] C. E. Chidume and C. O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, Journal of Mathematical Analysis and Applications, vol. 318 (2006) no. 1, 288-295.
- [20] Chidume, C.E., Mutangadura, S.A., An example on the Mann iteration method for Lipschitz pseudocontractions, Proc. Amer. Math. Soc., 129 (2001) No.8, 2359-2363.
- [21] F. Cianciaruso, G. Marino, A. Rugiano, B. Scardamaglia, On strong convergence of viscosity type method using averaged type mappings, J. Nonlinear Convex Anal., 16 (2015) 1619-1640.
- [22] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer, Dordrecht, 1990;
- [23] K. Deimling, Zeros of accretive operators, Manuscripta Math., 13 (1974) 365- 374.
- [24] F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point sets of nonexpansive mappings, Numer. Funct. Anal. Optim. 19 (1998) 33–56.
- [25] J. B. Diaz and F. T. Metcalf, On the set of subsequential limit points of successive approximations, Trans. Amer. Math. Soc. 135 (1969), 459-485.
- [26] W.G. Dotson, On the Mann iterative process, Trans. Amer. Math. Soc. 149 (1970) 65-73.
- [27] W. G. Dotson, Jr., Fixed points of quasi-nonexpansive mappings, J. Austral. Math. Soc. 13 (1972) 167-170.
- [28] M. Edelstein, A remark on a theorem of M. A. Krasnoselski, Amer. Math. Monthly 73 (1966) 509-510.
- [29] R. L. Franks and R. P. Marzec, A theorem on mean value iterations, Proc. Amer. Math. Soc. 30 (1971) 324-326.
- [30] J.Garcia-Falset, W. Kaczor, T. Kuczumow, S. Reich, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, Nonlinear Anal., 43 (2001) 377-401.
- [31] J. Garcia-Falset, Existence of fixed points for the sum of two operators, Math. Nachr. 283 (12) (2010) 1726-1757.
- [32] J. Garcia-Falset, K. Latrach, E. Moreno-Gálvez, M. A Taoudi, Shaefer-Krasnoselskii fixed points theorems using a usual measure of weak noncompactness, J. Diff. Equ. 352 (2012) 3436-3452.
- [33] J. Garcia-Falset, O. Muniz-Perez, Fixed point theory for 1-set contractive and pseudocontractive mappings, Applied Mathematics and Computation, 219 (2013), 6843-6855.
- [34] J. Garcia-Falset, G. Marino, R. Zaccone, An explicit midpoint algorithm in Banach spaces, J. Nonlinear and Convex Anal., 18 (2017) No: 11, 1933–1952.
- [35] A. Genel and J. Lindenstrauss, An example concerning fixed points, Israel J. Math., 22 (1975) 81-86.
- [36] M.K. Ghosh, L. Debnath, Convergence of Ishikawa iterates of quasi-nonexpansive mappings, J. Math. Anal. Appl., 207 (1997) 96-103.
- [37] D. Gohde, Zum Prinzip der kontraktiven Abbildung, (German) Math. Nachr., 30 (1965) 251-258.
- [38] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math., vol. 28, Cambridge Univ. Press, 1990.
- [39] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc., 73 (1967) 957-961 .
- [40] S. He, C. Yang , Solving the variational inequality problem defined on intersection of finite level sets, Abstr. Appl. Anal. Vol. 2013 (2013), Article ID 942315, 8 pages;
- [41] Z. He, W-S Du, Nonlinear algorithms approach to split common solution problems, Fixed Point Theory Appl. 2012, Article ID 130 (2012).
- [42] Hicks, T.L., Kubicek, J.R., On the Mann iteration process in Hilbert space, J. Math. Anal. Appl., 59 (1977) 498-504.
- [43] N. Hussain, G. Marino, L. Muglia, L. Alamri, On some Mann’s type iterative algorithms, Fixed Point Theory Appl 2015 (2015), Article ID 17;
- [44] S. Iemoto and W. Takahashi, Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space, Nonlinear Anal., 71 (2009) 2082-2089.
- [45] S. Ishikawa, Fixed points and iterations of a nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc. 59 (1976) 65-71.
- [46] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974) 147-150.

- [47] J.S. Jung, Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* 302 (2005) 509-520.
- [48] S. Kamimura, W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.*, 13 (2003) 938-945.
- [49] Y. Ke. C. Ma, The generalized viscosity implicit rules of nonexpansive mappings in Hilbert spaces, *Fixed Point Theory Appl.* 2015, 190 (2015), 21 pages;
- [50] T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal.*, 61 (2005) 51-60.
- [51] W. A. Kirk, A fixed point theorem for mappings which do not increase distance, *Amer. Math. Monthly*, 72 (1965) 1004-1006.
- [52] F. Kohsaka, W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, *Archiv der Mathematik*, 91 (2008) No. 2, 166-177.
- [53] M. A. Krasnoselskii, Two observations about the method of successive approximations, *Usp. Math. Nauk.*, 10(1955) 123-127.
- [54] K. Latrach, M. Aziz Taoudi, A. Zeghal, Some fixed point theorems of Schauder and the Krasnoselskii type and application to nonlinear transport equations, *J. Diff. Equ.*, 221 (2006) 256-271.
- [55] P. L. Lions, Approximation de points fixes de contractions, *C.R. Acad. Sci. Ser. A-B Paris*, 284 (1977) 1357-1359.
- [56] P.E. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.*, 16 (2008) 899-912.
- [57] P.E. Maingé, The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, *Computers and Mathematics with Applications*, 59 (2010) No. 1, 74-79.
- [58] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, 4 (1953) 506-510.
- [59] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.*, 318 (2006) No.1, 43-52.
- [60] G. Marino, B. Scardamaglia, R. Zaccone A general viscosity explicit midpoint rule for quasi-nonexpansive mappings, *J. Nonlinear and Convex Anal.*, 18 (2017) No.1, 137-148.
- [61] G. Marino, R. Zaccone, On strong convergence of some midpoint type methods for nonexpansive mappings, *J. Nonlinear Var. Anal.*, 1 (2017) No. 2, 159-174.
- [62] C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes, *Nonlinear Anal.*, 64 (2006) 2400-2411.
- [63] D. P. Milman, On some criteria for the regularity of spaces of the type (B) (Russian), *Dokl. Akad. Nauk SSSR* 20 (1938)
- [64] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, 241 (2000) 46-55.
- [65] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, 279 (2003) 372-379.
- [66] J.G. O'Hara, P. Pillay, H.K. Xu, Iterative approaches to finding nearest common fixed points of nonexpansive mappings in Hilbert spaces, *Nonlinear Anal.*, 54 (2003) 1417-1426.
- [67] E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer Series in Computational Mathematics, 2nd edn. (Springer-Verlag, 1993).
- [68] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, 73 (1967) 591-597.
- [69] W. V. Petryshyn, Construction of fixed points of demicompact mappings in Hilbert space, *Math. Anal. Appl.*, 14 (1966) 276-284.
- [70] B. J. Pettis, A proof that every uniformly convex space is reflexive, *Duke Math. J.* 5 (1939) no. 2, 249–253.
- [71] S. Reich, Asymptotic behavior of contractions in Banach spaces, *J. Math. Anal. Appl.* 44 (1973) 57-70.
- [72] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* 67 (1979) 274-276.
- [73] S. Reich, A limit theorem for projections, *Linear and Multilinear Algebra*, 13 (1983) no. 3, 281-290.
- [74] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.*, 75 (1980) 287-292.
- [75] H. Schaefer, Uber die Methode sukzessive Approximationen, *Jahre Deutsch Math. Verein* 59 (1957) 131-40.
- [76] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.*, 125 (1997) No.12, 3641-3645.
- [77] Y. Song and X. Chai, Halpern iteration for firmly type nonexpansive mappings, *Nonlinear Analysis: Theory, Methods And Applications*, 71(2009) No. 10, 4500-4506.
- [78] Y. Song, R. Chen, Viscosity approximation methods for nonexpansive nonself-mappings, *J. Math. Anal. Appl.*, 321 (2006) 316-326;
- [79] T. Suzuki, A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings, *Proceedings of the American Mathematical Society*, 135 (2007) No. 1, 99-106.
- [80] M.A. Taoudi, N. Salhi, B. Ghribi, Integrable solutions of a mixed type operator equation, *Appl. Math. Comput.*, 216 (2010) 1150-1157.
- [81] M. Tian, A general iterative algorithm for nonexpansive mappings in Hilbert spaces, *Nonlinear Anal.*, 73 (2010) 689-694.
- [82] F. Tricomi, Un teorema sulla convergenza delle successioni formate delle successive iterate di una funzione di una variabile reale, *Giorn. Mat. Battaglini*, 54 (1916) 1-9.
- [83] W. V. Petryshyn and T. E. Williamson, Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mapping, *J. Math. Anal. Appl.*, 43 (1973) 459-497.
- [84] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.*, 58 (1992) 486-491.

- [85] K. Wongchan and S. Saejung, On the strong convergence of viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, *Abstr. Appl. Anal.*, 2011 (2011), Article ID 385843, 9 pages
- [86] H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.*, 16 (1991) 1127-1138.
- [87] H.K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* 66 (2002) 240-256.
- [88] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bull. Austral. Math. Soc.* 65 (2002) 109-113.
- [89] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.* 298 (2004) 279–291.
- [90] H. K. Xu, T. H. Kim, X. Yin, Weak continuity of the normalized duality map, *Journal of Nonlinear and Convex Analysis*, 15 (2014) no 3, 595-604.
- [91] H. K. Xu, M. A. Alghamdi, N. Shahzad, The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces, *Fixed Point Theory Appl.* 41 (2015) 12 pages..
- [92] C. Yang, S. He, Convergence of explicit iterative algorithms for solving a class of variational inequalities, *Wseas Trans. Math.*, 13 (2014) 830-839;
- [93] H. Zhou, Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces, *Nonlinear Analysis*, 68 (2008) 2977-2983.