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Local convergence for a Chebyshev-type method in Banach space free of derivatives

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Abstract

This paper is devoted to the study of a Chebyshev-type method free of derivatives for solving nonlinear equations in Banach spaces. Using the idea of restricted convergence domain, we extended the applicability of the Chebyshev-type methods. Our convergence conditions are weaker than the conditions used in earlier studies. Therefore the applicability of the method is extended. Numerical examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

Keywords: Chebyshev-type method restricted convergence domain radius of convergence local convergence
2010 MSC: 65D10, 49M15, 74G20, 41A25

1. Introduction

Let $F : \Omega \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet differentiable operator between the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$. Due to the wide applications, finding a solution for equation

$$F(x) = 0 \tag{1}$$

is an important problem in applied mathematics and computational sciences. Convergence analysis of iterative methods require assumptions on the Fréchet derivatives of the operator F. That restricts the applicability of these methods.

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In this paper we study the seventh convergence order Chebyshev-type method [13]:

$$\begin{aligned} y_n &= x_n - A_n^{-1}F(x_n), \\ z_n &= y_n - B_nF(y_n), \\ x_{n+1} &= z_n - C_nF(z_n), \end{aligned} \tag{2}$$

where

$$\begin{aligned} A_n &= [w_n, x_n; F], \\ B_n &= (3I - A_n^{-1}([y_n, x_n; F] + [y_n, w_n; F]))A_n^{-1}, \\ C_n &= [z_n, x_n; F]^{-1}([w_n, x_n; F] + [y_n, x_n; F] - [z_n, x_n; F])A_n^{-1}, \\ w_n &= x_n + \gamma F(x_n), \ \ \gamma \in \mathbb{R}, \end{aligned}$$

$[.,.;F]$ denotes a divided difference of order one on Ω^2 and $x_0 \in \Omega$ is an initial point. Throughout this paper $L(\mathcal{B}_2, \mathcal{B}_1)$ denotes the set of bounded linear operators between $\mathcal{B}_1$ and $\mathcal{B}_2$.

The study of convergence of iterative algorithms is involving categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to derive conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to get estimates of the computed radii of the convergence balls. Local results are important since they tell us about the degree of difficulty in choosing initial points.

The above method was studied in [13]. Convergence analysis in [13] is based on the assumptions on the Fréchet derivative F up to the order seven. In this study, we use only assumptions on the first Fréchet derivative of the operator F in our convergence analysis, so the the method (2) can be applied to solve equations but the earlier results cannot be applied [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] (see Example 3.2).

The rest of the paper is structured as follows. In Section 2 we present the local convergence analysis of the method (2). We also provide a radius of convergence, computable error bounds and a uniqueness result. Numerical examples are given in the last section.

2. Local convergence

We need a definition concerning the monotonicity of functions.

Definition 2.1. *Let $T : D \subseteq \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function. We say T is nondecreasing on Ω, if for each $(a_1, a_2), (a_3, a_4) \in D$ with $a_1 \le a_3, a_2 \le a_4$,*

$$T(a_1, a_2) \le T(a_3, a_4). \tag{1}$$

Moreover, T is increasing on D, if $a_1 \le a_3$ and $a_2 < a_4$ or $a_1 < a_3$ and $a_2 \le a_4$ or $a_1 < a_3$ and $a_2 < a_4$ imply $T(a_1, a_2) < T(a_2, a_4)$.

Let us introduce some parameters and scalar functions to be used in the local convergence of method (2) that follows. Let $\gamma \in \mathbb{R}$ and $\delta \ge 0$ be parameters and let function $\omega_0 : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ be continuous and nondecreasing with $\omega_0(0, 0) = 0$. Define parameter r_0 by

$$r_0 = \sup\{t \in [0, +\infty) : \omega_0(\delta t, t) < 1\}. \tag{2}$$

Let $v_0 : [0, r_0) \longrightarrow [0, +\infty), \omega_1 : [0, r_0) \times [0, r_0) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions. Define functions g_1 and h_1 on the interval $[0, r_0)$ by

$$g_1(t) = \frac{\omega_1(|\gamma| v_0(t)t, t)}{1 - \omega_0(\delta t, t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

Suppose that

$$\omega_1(0,0) < 1. \tag{3}$$

Suppose that

$$h_1(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_0^-. \tag{4}$$

We have by (3) that

$$h_1(0) = \frac{\omega_1(0,0)}{1-\omega_0(0,0)} - 1 < 0. \tag{5}$$

Then, by (4), (5) and the intermediate value theorem equation $h_1(t) = 0$ has solutions in the interval $(0, r_0)$. Denote by r_1 the smallest such zero. Let $v : [0, r_0) \longrightarrow [0, +\infty), \omega_2 : [0, r_0) \longrightarrow [0, +\infty)$ and $\omega_3 : [0, r_0) \times [0, r_0) \longrightarrow [0, +\infty)$ be continuous and nondecreasing functions. Define functions β, g_2, h_2 on $[0, r_0)$ by

$$\beta(t) = \frac{1 + \omega_0(\delta t, t) + \omega_2((\delta + g_1(t)t)t) + \omega_3((\delta + g_1(t))t, |\gamma| v_0(t)t) v(g_1(t)t)}{(1-\omega_0(\delta t, t))^2}$$

$$g_2(t) = (1 + \beta(t) v(g_1(t)t)) g_1(t)$$

and

$$h_2(t) = g_2(t) - 1.$$

Suppose that

$$(1 + \beta(0)v(0))\omega_1(0,0) < 1 \tag{6}$$

and

$$h_2(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_0^- \tag{7}$$

We get by (6) that $h_2(0) < 0$. So, by the intermediate value theorem equation $h_2(t) = 0$ has solutions in the interval $(0, r_0)$. Denote by r_2 the smallest solution of $h_2(t) = 0$ in the interval $(0, r_0)$. Define functions p_1 and h_{p_1} on the interval $[0, r_0)$ by

$$p_1(t) = \omega_0(g_2(t)t, g_1(t)t)$$

and

$$h_{p_1}(t) = p_1(t) - 1.$$

We have by the definition of function w_0 that $h_{p_1}(0) < 0$. Suppose that

$$h_{p_1}(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_0^-. \tag{8}$$

Denote by r_{p_1} the smallest solution of equation $h_{p_1}(t) = 0$ on the interval $(0, r_0)$. Define functions φ, g_3, h_3 on the interval $[0, r_{p_1})$ by

$$\varphi(t) = \frac{1 + \omega_2((\delta + g_2(t)t) + \omega_0(g_2(t)t, t)}{(1 - p_1(t))(1 - \omega_0(\delta t, t))}$$

$$g_3(t) = (1 + \varphi(t) v(g_2(t)t)) g_2(t)$$

and

$$h_3(t) = g_3(t) - 1.$$

Suppose that

$$(1 + (1 + \omega_2(0))v(0))\,(1 + \beta(0)v(0))\omega_1(0,0) < 1, \tag{9}$$

and

$$h_3(t) \longrightarrow \text{ a positive number or } +\infty \text{ as } t \longrightarrow r_{p_1}^-. \tag{10}$$

We have that $h_3(0) < 0$. Denote by r_3 the smallest solution of equation $h_3(t) = 0$ in the interval $(0, r_0)$. Define the radius of convergence r by

$$r = \min\{r_i\} \ \ i = 1, 2, 3. \tag{11}$$

Then, for each $t \in [0, r)$

$$0 \le g_i(t) < 1 \tag{12}$$

$$0 \le p(t) < 1 \tag{13}$$

and

$$0 \le p_1(t) < 1. \tag{14}$$

Finally, define R^* by

$$R^* = \max\{r, \delta r\}. \tag{15}$$

Some alternatives to the aforementioned conditions are:
Equation

$$w_0(\delta t, t) = 1$$

has positive solutions. Denoted by r_0 the smallest such solution. Functions $v_0, \omega_1, v, \omega_2$ and ω_3 defined on the same intervals as before are increasing. Then, clearly conditions (4), (7), (8) and (10) hold.

We can show the local convergence analysis of method (2).

Theorem 2.2. *Let $F : \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a continuously Fréchet differentiable operator and let $[.,.;F] : \Omega \times \Omega \longrightarrow L(\mathcal{B}_1, \mathcal{B}_2)$be a divided difference of order one on $\Omega \times \Omega$ for F. Suppose: there exists $x^* \in \Omega$ and function $\omega_0 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ continuous and nondecreasing with $\omega_0(0, 0) = 0$ such that for each $x, y \in \Omega$,*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1); \tag{16}$$

and

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \le \omega_0(\|x - x^*\|, \|y - x^*\|). \tag{17}$$

Let $\Omega_0 = \Omega \cap B(x^*, r_0)$. There exist $\gamma \in \mathbb{R}, \delta \ge 0$, functions $v_0, v, \omega_2 : [0, r_0) \to [0, +\infty)$, $\omega_1, \omega_3 : [0, r_0) \times [0, r_0) \to [0, +\infty)$ such that for each $x, y, z \in \Omega_0$

$$\|I + \gamma[x, x^*; F]\| \le \delta, \tag{18}$$

$$\|[x, x^*; F]\| \le v_0(\|x - x^*\|), \tag{19}$$

$$\|F'(x^*)^{-1}[x, x^*; F]\| \le v(\|x - x^*\|), \tag{20}$$

$$\|F'(x^*)^{-1}([x, y; F] - [y, x^*; F])\| \le \omega_1(\|x - y\|, \|y - x^*\|), \tag{21}$$

$$\|F'(x^*)^{-1}([x, y; F] - [z, y; F])\| \le \omega_2(\|x - z\|), \tag{22}$$

$$\|F'(x^*)^{-1}([x, y; F] - [z, x; F])\| \le \omega_3(\|x - z\|, \|y - x\|), \tag{23}$$

$$\bar{B}(x^*, R^*) \subseteq \Omega, \tag{24}$$

(4), (7), (8) and (9) hold. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method (2) is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \le g_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \tag{25}$$

$$\|z_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| \tag{26}$$

and

$$\|x_{n+1} - x^*\| \le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|, \tag{27}$$

where the functions $g_i, i = 1, 2, 3$ are defined previously. Furthermore, if there exists for $R_1 \ge r$ such that

$$\omega_0(R_1, 0) < 1 \ \text{ or } \ \omega_0(0, R_1) < 1, \tag{28}$$

then the limit point x^* is the only solution of equation $F(x) = 0$ in $\Omega_1 := \Omega \cap B(x^*, R_1)$.

Proof. The proof is induction based. By hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, the definition of w_0, A_0, r the fact that ω_0 is nondecreasing, we have that

$$\begin{aligned}
& \|F'(x^*)^{-1}(A_0 - F'(x^*))\| \\
\leq\ & (\text{by } (17))\, \omega_0(\|w_0 - x^*\|, \|x_0 - x^*\|) \\
\leq\ & (\text{by } (2))\, \omega_0(\|x_0 - x^* + \gamma[x_0, x^*; F](x_0 - x^*)\|, \|x_0 - x^*\|) \\
\leq\ & \omega_0(\|(I + \gamma[x_0, x^*; F])(x_0 - x^*)\|, \|x_0 - x^*\|) \\
\leq\ & (\text{by } (1) \text{ and } (2))\, \omega_0(\delta r, r) < 1.
\end{aligned} \tag{29}$$

In view of (29) and the Banach perturbation lemma [2, 3], we get that A_0 is invertible and

$$\|A_0^{-1} F'(x^*)\| \leq \frac{1}{1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|)}. \tag{30}$$

We also have that y_0 is well defined by the first substep of method (2) for $n = 0$. We can write by method (2) and (16) that

$$\begin{aligned}
y_0 - x^* &= (\text{by } (2))\, x_0 - x^* - A_0^{-1} F(x_0) \\
&= (\text{by } (10))\, A_0^{-1}(A_0(x_0 - x^*) - [x_0, x^*; F](x_0 - x^*)) \\
&= A_0^{-1} F'(x^*)[F'(x^*)^{-1}([u_0, x_0; F] - [x_0, x^*; F])](x_0 - x^*).
\end{aligned} \tag{31}$$

By the first substep of method (2) for $n = 0$, the definition of r, g_1, the fact that w_1 is nondecreasing, we obtain in turn that

$$\begin{aligned}
& \|y_0 - x^*\| \\
\leq\ & (\text{by } (2))\, \|A_0^{-1} F'(x^*)\| \|F'(x^*)^{-1}([w_0, x_0; F] - [x_0, x^*; F])\| \|x_0 - x^*\| \\
\leq\ & (\text{by } (21) \text{ and } (30))\, \frac{\omega_1(\|w_0 - x_0\|, \|x_0 - x^*\|)\|x_0 - x^*\|}{1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|)} \\
\leq\ & (\text{by } (2) \text{ and } (19))\, \frac{\omega_1(|\gamma| v_0(\|x_0 - x_0\|)\|x_0 - x^*\|}{1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|)} \|x_0 - x^*\| \\
=\ & g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq (\text{by } (7) \text{ for } i = 1)\, \|x_0 - x^*\| < r,
\end{aligned} \tag{32}$$

which shows (25) for $n = 0$ and $y_0 \in B(x^*, r)$. We need an estimate on $\|B_0 F'(x^*)\|$. By the definition of B_0, β and the fact that functions $\omega_0, \omega_2, \omega_3$ are nondecreasing, we have in turn that

$$\begin{aligned}
& \|B_0 F'(x^*)\| \\
=\ & \|A_0^{-1}(3A_0 - [y_0, x_0; F] - [y_0, w_0; F])A_0^{-1}\| \\
\leq\ & \|A_0^{-1} F'(x^*)\|^2 [\|F'(x^*)^{-1} F'(x^*)\| \\
& + \|F'(x^*)^{-1}([w_0, w_0; F] - F'(x^*))\| \\
& + \|F'(x^*)^{-1}([w_0, x_0; F] - [y_0, x_0; F])\| \\
& + \|F'(x^*)^{-1}([w_0, x_0; F] - [y_0, w_0; F])\|] \\
\leq\ & (\text{by } (22), (23, (32)) \\
& \frac{1 + \omega_0(\|x_0 - x^*\|, \|x_0 - x^*\|) + \omega_2(\|w_0 - y_0\|) + \omega_3(\|w_0 - y_0\|, \|x_0 - w_0\|)}{(1 - \omega_0(\delta\|x_0 - x^*\|, \|x_0 - x^*\|))^2} \\
\leq\ & \beta(\|x_0 - x^*\|).
\end{aligned} \tag{33}$$

By the second substep of method (2), the fact that function v is nondecreasing , β is nonnegative and the definition of g_2 we get in turn that

$$\begin{aligned}
&\|z_0 - x^*\| \\
\le\ & \text{(by the triangle inequality) } \|y_0 - x^*\| + \|B_0 F'(x^*)\| \|F'(x^*)^{-1} F(y_0)\| \\
\le\ & \text{(by (33)) } (1 + \beta(\|y_0 - x^*\|) v(\|y_0 - x^*\|))\, \|y_0 - x^*\| \\
\le\ & \text{(by (32) and (33))} \\
& (1 + \beta(\|x_0 - x^*\|) v(g_1(\|x_0 - x^*\| \|x_0 - x^*\|))\, g_1(\|x_0 - x^*\|) \|x_0 - x^*\| \\
=\ & \text{(by the definition of function } g_2) \, g_2(\|x_0 - x^*\|) \|x_0 - x^*\| \\
\le\ & \text{(by (12) (for i=2)) } \|x_0 - x^*\| < r, \qquad (34)
\end{aligned}$$

which shows (26) for $n = 0$ and $z_0 \in B(x^*, r)$. We must show $[z_0, y_0; F]^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$. We get that

$$\begin{aligned}
&\|F'(x^*)^{-1}([z_0, y_0; F] - F'(x^*))\| \\
\le\ & \text{(by (17)) } \omega_0(\|z_0 - x^*\|, \|y_0 - x^*\|) \\
\le\ & \text{(by (32) and (34)) } \omega_0(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|, g_1(\|x_0 - x^*\|)\|x_0 - x^*\|) \\
=\ & \text{(by the definition of function } p_1) \, p_1(\|x_0 - x^*\|) \qquad (35) \\
\le\ & \text{(by (14)) } p_1(r) < 1,
\end{aligned}$$

so

$$\|[z_0, y_0; F]^{-1} F'(x^*)\| \le \frac{1}{1 - p_1(\|x_0 - x^*\|)}. \qquad (36)$$

To obtain an estimate on $\|C_0 F'(x^*)\|$,

$$\begin{aligned}
&\|F'(x^*)^{-1}(([w_0, x_0; F] - [z_0, x_0; F]) + (]y_0, x_0; F] - F'(x^*)) + F'(x^*))\| \\
\le\ & \text{(by (17) and (22)) } 1 + \omega_2(\|w_0 - z_0\|) + \omega_0(\|y_0 - x^*\|, \|x_0 - x^*\|) \\
\le\ & \text{(by the triangle inequality)} \\
& 1 + \omega_2(\|w_0 - x^*\| + \|z_0 - x^*\|) + \omega_0(\|y_0 - x^*\|, \|x_0 - x^*\|),
\end{aligned}$$

so by the definition of φ

$$\begin{aligned}
\|C_0 F'(x^*)\| \le\ & \text{(by (31) and (36))} \\
& \frac{1 + \omega_2(\|w_0 - x^*\|, \|z_0 - x^*\|) + \omega_0(\|y_0 - x^*\|, \|x_0 - x^*\|)}{(1 - p_1(\|x_0 - x^*\|))(1 - \omega_0 \delta(\|x_0 - x^*\|, \|x_0 - x^*\|))} \\
\le\ & \varphi(\|x_0 - x^*\|) \qquad (37)
\end{aligned}$$

leading by the third substep of method (2) (by (11), (12) (for $i = 2$), and (37)) to the estimate

$$\begin{aligned}
&\|x_1 - x^*\| \\
\le\ & \text{(by the triangle inequality) } \|z_0 - x^*\| + \|C_0 F'(x^*)\| \|F'(x^*)^{-1} F(z_0)\| \\
\le\ & \text{(by (20) and (37)) } (1 + \varphi(\|x_0 - x^*\|) v(\|z_0 - x^*\|)) \|z_0 - x^*\| \\
\le\ & \text{(by (34)) } (1 + \varphi(\|x_0 - x^*\|) v(g_2(\|x_0 - x^*\|)\|x_0 - x^*\|) \\
& \times g_2(\|x_0 - x^*\|) \|x_0 - x^*\| \\
=\ & \text{(by the definition of } g_3) \, g_3(\|x_0 - x^*\|) \|x_0 - x^*\| \qquad (38) \\
\le\ & \text{(by (12) for } i = 3) \, \|x_0 - x^*\| < r,
\end{aligned}$$

which shows (27) and $x_1 \in U(x^*, r)$. The induction for (25)– (27) is completed in an analogous way, if we replace x_0, y_0, z_0, u_0, x_1 by $x_k, y_k, z_k, u_k, x_{k+1}$, respectively, in the previous estimates. Then, it follows from the estimate

$$\|x_{k+1} - x^*\| \le c \|x_k - x^*\| < r, \qquad (39)$$

where $c = g_3(\|x_0 - x^*\|) \in [0, 1)$, that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. Let $y^* \in \Omega_1$ with $F(y^*) = 0$. Define Q by $Q = [y^*, x^*; f]$. Then, we get that

$$\begin{aligned} \|F'(x^*)^{-1}(Q - F'(x^*))\| &\leq \text{(by (17))}\, \omega_0(0, \|y^* - x^*\|) \\ &\leq \text{(by (28))}\, \omega_0(0, R_1) < 1, \end{aligned} \tag{40}$$

so Q is invertible. Then, from the identity $0 = F(y^*) - F(x^*) = Q(y^* - x^*)$, we conclude that $x^* = y^*$.

□

Remark 2.3. *Method (2) is not changing if we use the new instead of the old conditions [13]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC) [14]*

$$\xi = \frac{ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } \; n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.2.

3. Numerical Examples

The numerical examples are presented in this section. We choose

$$[x, y; F] = \int_0^1 F'(y + \theta(x - y))d\theta.$$

Example 3.1. *Let $X = \mathbb{R}^3, \Omega = \bar{U}(0, 1), x^* = (0, 0, 0)^T$. Define function F on Ω for $q = (x, y, z)^T$ by*

$$F(q) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(q) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (18)-(23) conditions, we get $\omega_0(s, t) = \frac{L_0}{2}(s+t), \omega_1(s, t) = \frac{Ls+L_0t}{2}, \omega_2(t) = \frac{1}{2}e^{\frac{1}{L_0}}t, \omega_3(s, t) = \frac{L}{2}(s+t), v_0(t) = v(t) = \frac{1}{2}(1+e^{\frac{1}{L_0}}), r_0 = \frac{1}{L_0}, \delta = 1+\frac{1}{2}|\gamma|(1+e^{\frac{1}{L_0}}), L_0 = e-1$ and $L = e$. The parameters are

$$r_1 = 0.2010,\ r_2 = 0.0830,\ r_3 = 0.0639 = r.$$

Example 3.2. *Let $X = C[0, 1], \Omega = \bar{B}(x^*, 1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [7, 11] defined by*

$$x(s) = \int_0^1 K(s, t)\frac{x(t)^2}{2}dt,$$

where the kernel K is the Green's function defined on the interval $[0, 1] \times [0, 1]$ by

$$K(s, t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 K(s,t)\frac{x(t)^2}{2}dt.$$

Notice that [5, 7, 8]

$$\| \int_0^1 K(s,t)dt\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 K(s,t)x(t)dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\|x - y\|.$$

We can choose $\omega_0(t,s) = \omega_1(t,s) = \omega_3(s,t) = \frac{t+s}{16}$, $\omega_2(t) = \frac{1}{16}t, v(t) = \frac{9}{16}$ and $\delta = 1 + |\gamma|\frac{9}{16}$. The parameters are

$$r_1 = 0.5805,\ r_2 = 0.2623,\ r_3 = 0.1463 = r.$$

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