A new Generalization of Wardowski Fixed Point Theorem in Complete Metric Spaces

Andreea Fulga\textsuperscript{a}, Alexandrina Proca\textsuperscript{a}

\textsuperscript{a} Department of Mathematics and Computer Sciences, Transilvania University of Brasov, Brasov, Romania.

Abstract

The aim of this paper is to state and prove Wardowski type fixed point theorem in metric spaces. The paper includes an example which shows that our result is a proper extension of some known results.

Keywords: Wardowski type contraction, fixed point, metric space

2010 MSC: 54A05, 54C60.

1. Introduction and Preliminaries

Starting from one of the fundamental results of fixed point theory known as the Banach contraction principle [5], several authors proved many interesting extensions and generalizations ([1]-[4], [6]-[18]).

In 2012, D. Wardowski [14], using functions $F: \mathbb{R}_+ \to \mathbb{R}$ proved a fixed point theorem concerning a new type of contractions, called $F$–contractions.

Let function $F: \mathbb{R}_+ \to \mathbb{R}$ such that:

\begin{itemize}
  \item [(F1)] $F$ is strictly increasing, that is, for all $x, y \in \mathbb{R}_+$ if $x < y$ then $F(x) < F(y)$;
  \item [(F2)] For each sequence $\{\alpha_n\}$ of positive numbers,
    \[
    \lim_{n \to \infty} \alpha_n = 0 \text{ if only if } \lim_{n \to \infty} F(\alpha_n) = -\infty;
    \]
  \item [(F3)] There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} (\alpha^k F(\alpha)) = 0$
\end{itemize}

We denote by $\mathcal{F}$ the family of all that functions.

Email addresses: afulga@unitbv.ro (Andreea Fulga), alexproca@unitbv.ro (Alexandrina Proca)

Received July 14, 2017, Accepted: August 12, 2017, Online: August 19, 2017.
Definition 1.1. [14] Let $(X,d)$ be a metric space. A map $T : X \to X$ is said to be an $F$–contraction on $(X,d)$ if there exists $F \in F$ and $\tau > 0$ such that for all $x,y \in X$

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))$$

(1)

Theorem 1.2. [14] Let $(X,d)$ be a complete metric space and $T : X \to X$ be an $F$–contraction. Then $T$ has a unique fixed point $x^*$ and for all $x \in X$ the sequence $\{T^n x\}$ is convergent to $x^*$.

Remark 1.3. From (F1) and (1) it follows that

$$F(d(Tx,Ty)) \leq F(d(x,y)) - \tau < F(d(x,y))$$

for all $x,y \in X$ such that $Tx \neq Ty$. Also, $T$ is a continuous operator.

Afterwards, Wardowski and Van Dung [15] have introduced the notion of a $F$–weak contraction, in this way.

Definition 1.4. [15] Let $(X,d)$ be a metric space. A map $T : X \to X$ is said to be a $F$–weak contraction on $(X,d)$ if there exists $F \in F$ and $\tau > 0$ such that for all $x,y \in X$ satisfying $d(Tx,Ty) > 0$, the following holds:

$$\tau + F(d(Tx,Ty)) \leq F(M(x,y))$$

(2)

where

$$M(x,y) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2} \right\}.$$

By using this notion, Wardowski and Van Dung [15] have demonstrated a fixed point theorem which generalizes the theorem 1.2 as follows.

Theorem 1.5. [15] Let $(X,d)$ be a complete metric space and $T : X \to X$ be a $F$–weak contraction. If $T$ or $F$ is continuous, then $T$ has a unique fixed point $x^*$ and for all $x \in X$ the sequence $\{T^n x\}$ is convergent to $x^*$.

Latter, Piri and Kumam [12] introduced a large class of functions by replacing the condition $(F3)$ in the definition of $F$–contraction with the following

$(F3')$ F is continuous on $(0, \infty)$

and they denote the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfies the conditions $(F1), (F2)$, and $(F3')$ by $\mathfrak{F}$.

With this assumptions, Piri and Kumam [12] proved the next fixed point theorem.

Theorem 1.6. [12] Let $(X,d)$ be a complete metric space and a mapping $T : X \to X$. Suppose there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that, for all $x,y \in X$

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)).$$

Then $T$ has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}$ converges to $x^*$.

In this paper, using the idea from [10], we introduce a new type of $F$–contraction, and prove a fixed point theorem which generalizes some known results.
2. Main results

First, let $F_E$ denote the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfies the following conditions:

$(F_E 1)$ $F$ is strictly increasing, that is, for all $x, y \in \mathbb{R}_+$, if $x < y$ then $F(x) < F(y)$;

$(F_E 2)$ There exists $\tau > 0$ such that $\tau + \lim_{t \to t_0} \inf F(t) > \lim_{t \to t_0} \sup F(t)$, for every $t_0 > 0$.

**Definition 2.1.** Let $(X, d)$ be a metric space. A map $T : X \to X$ is said to be a $F_E$-contraction on $(X, d)$ if there exists $F \in F_E$ and $\tau > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(E(x, y))$$

where

$$E(x, y) = d(x, y) + |d(x, Tx) - d(y, Ty)|.$$  \hfill (4)

**Remark 2.2.** (1) Every $F_E$-contraction is an $F$-contraction, but the inverse implication does not hold.

(2) Not every $F$-weak contraction is a $F_E$-contraction.

The following example shows that the statements from previous remark hold.

**Example 2.3.** Let $X = [0, \frac{7}{10}] \cup \{1\}$ and $d(x, y) = |x - y|$, $x, y \in X$. Then $(X, d)$ is a complete metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq \frac{7}{10} \\ \frac{1}{4}, & x = 1 \end{cases}$$

and choosing $F(\alpha) = \ln \alpha$, $\alpha \in (0, \infty)$ and $\tau = \ln 7$.

Since $T$ is not continuous, $T$ is not an $F$-contraction. In addition to that, for $x = \frac{1}{4}$ and $y = 1$ we have

$$d\left(T\frac{1}{4}, T1\right) = \left|\frac{1}{8} - \frac{1}{4}\right| = \frac{1}{8} > 0$$

and

$$M\left(\frac{1}{4}, 1\right) = \max\left\{d\left(\frac{1}{4}, 1\right), d\left(\frac{1}{4}, T\frac{1}{4}\right), d\left(1, T1\right), \frac{d\left(1, T\frac{1}{4}\right) + d\left(\frac{1}{4}, T1\right)}{2}\right\}$$

$$= \max\left\{\frac{1}{8}, \frac{3}{4}, \frac{7}{16}\right\} = \frac{3}{4}.$$

Then,

$$\tau + F\left(d\left(T\frac{1}{4}, T1\right)\right) = \ln 7 + \ln \left(\frac{1}{8}\right) = \ln \left(\frac{7}{8}\right)$$

$$\geq \ln \left(\frac{3}{4}\right) = F\left(M\left(\frac{1}{4}, 1\right)\right)$$

so $T$ is not a $F$-weak contraction.

For $x \in [0, \frac{7}{10}]$ and $y = 1$, we have

$$d(Tx, T1) = d\left(\frac{x}{2}, \frac{1}{4}\right) = \left|\frac{2x - 1}{4}\right|$$

and

$$E(x, 1) = d(x, 1) + |d(x, Tx) - d(1, T1)|$$

$$= 1 - x + \left|\frac{x}{2} - \frac{3}{4}\right| = \left|\frac{7 - 6x}{4}\right|.$$
Therefore,

\[ \ln 7 + \ln (d(Tx, T1)) \leq \ln (E(x, 1)) \iff \ln \left( \frac{2x - 1}{4} \right) \leq \ln \left( \frac{7 - 6x}{4} \right) \iff 7 \cdot \frac{2x - 1}{4} \leq 7 - 6x. \]

For \( x \leq \frac{1}{2} \),

\[ 7 \cdot \frac{2x - 1}{4} \leq 7 - 6x \iff 7 - 14x \leq 7 - 6x \iff x \geq 0, \]

and for \( x > \frac{1}{2} \)

\[ 7 \cdot \frac{2x - 1}{4} \leq 7 - 6x \iff 14x - 7 \leq 7 - 6x \iff x \leq \frac{7}{10} \]

which prove that \( T \) is a \( F_E \)-contraction.

Now we state the main result of the paper.

**Theorem 2.4.** Let \((X, d)\) be a complete metric space and \( T : X \to X \) be a \( F_E \)-contraction. Then \( T \) has a unique fixed point \( x^* \) and for all \( x_0 \in X \) the sequence \( \{T^n x_0\} \) is convergent to \( x^* \).

**Proof.** Let \( x_0 \in X \) be arbitrary and fixed and we define \( x_{n+1} = Tx_n = T^n x_0 \) for all \( n \in \mathbb{N} \). If there exists \( n_0 \in \mathbb{N} \cup \{0\} \) such that \( x_{n_0+1} = x_{n_0} \), because \( x_{n_0+1} = Tx_{n_0} \), we obtain that \( Tx_{n_0} = x_{n_0} \), so \( x_{n_0} \) is a fixed point of \( T \).

Now, we suppose that \( x_{n+1} \neq x_n \) for all \( n \in \mathbb{N} \cup \{0\} \). So, \( d(x_n, x_{n+1}) > 0 \), \( (\forall) n \in \mathbb{N} \cup \{0\} \) and from (3) it follows that, for all \( n \in \mathbb{N} \)

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) > 0 \Rightarrow \\
\Rightarrow \tau + F(d(Tx_{n-1}, Tx_n)) \leq F(E(x_{n-1}, x_n)) \\
\Rightarrow \tau + F(d(x_n, x_{n+1})) \leq \\
\leq F(d(x_{n-1}, x_n) + |d(x_{n-1}, Tx_{n-1}) - d(x_n, Tx_n)|) \iff \\
\tau + F(d(x_n, x_{n+1})) \leq \\
\leq F(d(x_{n-1}, x_n) + |d(x_{n-1}, x_n) - d(x_n, x_{n+1})|
\]

or, if we denote by \( d_n = d(x_{n-1}, x_n) \), we have

\[ \tau + F(d_{n+1}) \leq F(d_n + |d_n - d_{n+1}|). \quad (5) \]

If there exists \( n \in \mathbb{N} \) such that \( d_{n+1} \geq d_n \), then (5) becomes

\[ \tau + F(d_{n+1}) \leq F(d_{n+1}) \Rightarrow \tau \leq 0. \]

But, this is a contradiction, so, for \( d_{n+1} < d_n \) we have

\[ \tau + F(d_{n+1}) \leq F(2d_n - d_{n+1}) \quad (6) \]

\[ \Rightarrow F(d_{n+1}) \leq F(2d_n - d_{n+1}) - \tau < F(2d_n - d_{n+1}) \]

and using \((F_E)\)

\[ d_{n+1} < 2d_n - d_{n+1}. \]

Therefore, the sequence \( \{d_n\} \) is strictly increasing and bounded.
Now, let \( d = \lim_{n \to \infty} d_n \) and we suppose that \( d > 0 \). Because \( d_n \searrow d \) it results that \( (2d_n - d_{n+1}) \searrow d \) and taking the limit as \( n \to \infty \) in (6), we get
\[
\tau + F(d + 0) \leq F(d + 0) \Rightarrow \tau \leq 0.
\]
It is a contradiction, so
\[
d = \lim_{n \to \infty} d_n = \lim_{n \to \infty} d(x_{n-1}, x_n) = 0. \tag{7}
\]
In order to prove that \( \{x_n\} \) is a Cauchy sequence in \( (X, d) \), we suppose the contrary, that is, there exists \( \varepsilon > 0 \) and the sequences \( \{n(k)\}, \{m(k)\} \) of positive integers, with \( n(k) > m(k) > k \) such that
\[
d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \quad \text{and} \quad d(x_{n(k)-1}, x_{m(k)}) < \varepsilon \tag{8}
\]
for any \( k \in \mathbb{N} \).

Then, we have
\[
\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)})
\]
\[
< d(x_{n(k)}, x_{n(k)-1}) + \varepsilon.
\]
Letting \( k \to \infty \) and using (7) it follows
\[
\lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{9}
\]
Furthermore, using the triangle inequality, we obtain that
\[
0 \leq |d(x_{n(k)+1}, x_{m(k)+1}) - d(x_{n(k)}, x_{m(k)})|
\]
\[
= d(x_{n(k)+1}, x_{n(k)}) + d(x_{m(k)}, x_{m(k)+1})
\]
and
\[
\lim_{k \to \infty} |d(x_{n(k)+1}, x_{m(k)+1}) - d(x_{n(k)}, x_{m(k)})|
\]
\[
= \lim_{k \to \infty} [d(x_{n(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1})] = 0.
\]
So,
\[
\lim_{k \to \infty} d(x_{n(k)+1}, x_{m(k)+1}) = \lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{10}
\]
On the other hand, because from (7)
\[
\lim_{n \to \infty} d(x_n, Tx_n) = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0,
\]
there exists \( N \in \mathbb{N} \) such that
\[
d(x_{n(k)}, Tx_{n(k)}) < \frac{\varepsilon}{4} \quad \text{and} \quad d(x_{m(k)}, Tx_{m(k)}) < \frac{\varepsilon}{4}, \quad (\forall) \ k \geq N. \tag{11}
\]
Assuming by contradiction, that there exists \( l \in \mathbb{N} \) such that \( d(x_{n(l)+1}, x_{m(l)+1}) = 0 \), from (11) and (7) it follows that
\[
\varepsilon \leq d(x_{n(l)}, x_{m(l)})
\]
\[
\leq d(x_{n(l)}, x_{n(l)+1}) + d(x_{n(l)+1}, x_{m(l)+1}) + d(x_{m(l)+1}, x_{m(l)})
\]
\[
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\]
This is a contradiction. So we proved that the inequality occurs
\[
d(Tx_{n(k)}, Tx_{m(k)}) = d(x_{n(k)+1}, x_{m(k)+1}) > 0 \tag{12}
\]
for all $k \geq N$, and using (3), there exists $\tau > 0$ such that

$$
\tau + F\left(d(Tx_{m(k)}, Tm(k))\right) \leq F\left(E(x_{n(k)}, x_{m(k)})\right)
$$

for any $k$, where

$$
E(x_{n(k)}, x_{m(k)}) = d(x_{n(k)}, x_{m(k)}) + \left|d(x_{n(k)}, Tx_{m(k)}) - d(x_{m(k)}, Tm(k))\right|
$$

Hence $\lim_{k \to \infty} E(x_{n(k)}, x_{m(k)}) = \varepsilon$ and by (10) we have

$$
\tau + \lim_{k \to \infty} \inf_{k \to \infty} F\left(d(Tx_{n(k)}, Tm(k))\right) \leq \lim_{k \to \infty} \inf_{k \to \infty} F\left(E(x_{n(k)}, x_{m(k)})\right) \\
\leq \lim_{k \to \infty} \sup_{k \to \infty} F\left(E(x_{n(k)}, x_{m(k)})\right) \\
\Leftrightarrow \tau + F(\varepsilon+) \leq F(\varepsilon+)
$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and by completeness of $X$ there converges to some point $x^* \in X$.

Next, we show that $x^*$ is a fixed point of $T$. We consider two cases:

1. For any $n \in \mathbb{N}$ there exists $k_n > k_{n-1}$, $k_0 = 1$ and $x_{k_{n+1}} = Tx^*$. Then, $x^* = \lim_{n \to \infty} x_{k_{n+1}} = Tx^*$, so $x^*$ is fixed point of $T$.
2. There exists $m \in \mathbb{N}$ such that for all $n \geq m$, $d(Tx_n, Tx^*) > 0$. Substituting $x = x_n$ and $y = x^*$ in (3), there exists $\tau > 0$ such that

$$
\tau + F(d(Tx_n, Tx^*)) \leq F(E(x_n, x^*)) \\
\tau + F(d(x_{n+1}, x^*)) \leq F(d(x_n, x^*) + d(x_n, Tx_n) - d(x^*, Tx^*)) \\
\tau + F(d(x_{n+1}, x^*)) \leq F(d(x_n, x^*) + d(x_n, x_{n+1}) - d(x^*, Tx^*)).
$$

We suppose that $x^* \neq Tx^*$. Letting $n \to \infty$, from (7) we obtain

$$
\tau + \lim_{t \to d(x^*, Tx^*)} F(t) < \lim_{t \to d(x^*, Tx^*)} F(t) < \lim_{t \to d(x^*, Tx^*)} \sup F(t)
$$

which contradicts ($F_E$2) of the hypothesis. Hence $Tx^* = x^*$.

Now, let us show that $T$ must have only one fixed point. If there exists another point $y^* \in X$, $x^* = y^*$ such that $Ty^* = y^*$, then $d(x^*, y^*) = d(Tx^*, Ty^*) > 0$ and we get

$$
\tau + F(d(Tx^*, Ty^*)) \leq F(E(x^*, y^*)) \\
\tau + F(d(x^*, y^*)) \leq F(d(x^*, y^*) + d(x^*, Tx^*) - d(y^*, Ty^*)) \\
\tau + F(d(x^*, y^*)) \leq F(d(x^*, y^*) + d(x^*, x^*) - d(y^*, y^*)) \\
\tau + F(d(x^*, y^*)) \leq F(d(x^*, y^*))
$$

which is a contradiction.

**Example 2.5.** Let $T$ be given as in Example 2.3. Since $T$ is not a contraction, Theorem 1.2 is not applicable to $T$ and because $T$ is not a $F$-weak contraction, Theorem 1.6 cannot be applied. On the other hand let $F$ and $\tau$ be given as in Example 2.3. Then $T$ is an $F_E$ contraction, and Theorem 2.4 can be applicable to $T$ and the unique fixed point of $T$ is 0.
References


Andreea Fulga
Department of Mathematics, Transilvania University of Brasov,
Brasov, Romania.
e-mail: afulga@unitbv.ro

Alexandrina Proca
Department of Mathematics, Transilvania University of Brasov,
Brasov, Romania.
e-mail:alexproca@unitbv.ro