Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle

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Abstract


Keywords: Space of continuous function, operator with Volterra property, max-norm, Bielecki norm, contraction, $G$-contraction, fiber contraction, progressive contraction, step by step contraction, fixed point, Picard operator, weakly Picard operator, differential equation, integral equation, conjecture. 2010 MSC: 47H10, 47H09, 34K05, 34K12, 45D05, 45G10, 54H25.

1. Introduction

Following an idea of T.A. Burton ([7], [8], [9], ...) of progressive contractions, and the forward step method ([21]), in this paper we give some variants of contraction principle in the case of operators with...
Volterra property. The basic ingredient in our variant, step by step contraction principle, is \(G\)-contraction \([20]\). Some applications to differential and integral equations are also given. In connection with our abstract results, a conjecture is formulated.

2. Preliminaries

2.1. \(G\)-contractions

Let \((X,d)\) be a metric space and \(G \subset X \times X\) be a nonempty subset. An operator \(f : X \to X\) is a \(G\)-contraction if there exists \(l \in ]0,1[\) such that,

\[
d(f(x), f(y)) \leq ld(x,y), \forall (x,y) \in G.
\]

Here are some examples of subsets \(G \subset X \times X\):

1. \(G := G(f)\), the graphic of the operator \(f\). In this case, a \(G\)-contraction is a graphic contraction \([17, 24, \ldots]\).

2. Let \(A_i \subset X\), \(i = 1, p\), be nonempty closed subsets such that:

   (i) \(X = \bigcup_{i=1}^{p} A_i\); 

   (ii) \(f(A_i) \subset A_{i+1}\), \(i = 1, p\), \((A_{p+1} = A_1)\).

For, \(G := \bigcup_{i=1}^{p} (A_i \times A_{i+1})\), a \(G\)-contraction is a cyclic contraction of Kirk-Srinivasan-Veeramani (see the references in \([20]\)).

3. Let \(a, b, c \in \mathbb{R}\), \(a < c < b\) and \(X := C[a,b]\) with \(d(x,y) := \max_{t \in [a,b]} |x(t) - y(t)|\). For \(K, H \in C([a,b] \times [a,b] \times \mathbb{R}, \mathbb{R})\), we consider the operator, \(f : C[a,b] \to C[a,b]\), defined by,

\[
f(x)(t) := \int_{a}^{c} K(t,s,x(s))ds + \int_{a}^{t} H(t,s,x(s))ds, \ t \in [a,b].
\]

We suppose that there exists \(L_H > 0\) such that

\[
|H(t,s,u) - H(t,s,v)| \leq L_H |u - v|, \ \forall \ t, s \in [a,b], \ \forall \ u, v \in \mathbb{R}.
\]

If, \(L_H (b - c) < 1\) and if we take

\[
G := \{(x,y) \in C[a,b] \times C[a,b] \ | \ x|_{[a,c]} = y|_{[a,c]}\},
\]

then \(f\) is a \(G\)-contraction.

For other examples of \(G\)-contractions see \([20]\) and \([24]\), pp. 282-284.

2.2. Weakly Picard operators

Let \((X,\to)\) be an \(L\)-space \(((X,d), \to; (X,\tau), \to; (X,\|\cdot\|), \to; \ldots)\). An operator \(f : X \to X\) is weakly Picard operator \((WPO)\) if the sequence, \((f^n(x))_{n \in \mathbb{N}}\), converges for all \(x \in X\) and the limit (which generally depend on \(x\)) is a fixed point of \(f\).

If an operator \(f\) is \(WPO\) and the fixed point set of \(f\), \(F_f = \{x^*\}\), then by definition \(f\) is Picard operator \((PO)\).

For a \(WPO\), \(f : X \to X\), we define the operator \(f^\infty : X \to X\), by \(f^\infty(x) := \lim_{n \to \infty} f^n(x)\).

We remark that, \(f^\infty(X) = F_f\), i.e., \(f^\infty\) is a set retraction of \(X\) on \(F_f\).

For the case of ordered \(L\)-spaces, we have some properties of \(WPO\) and \(PO\).

Abstract Gronwall Lemma. Let \((X,\to, \leq)\) be an ordered \(L\)-space and \(f : X \to X\) be an operator. We suppose that:
\( f \) is increasing;

(2) \( f \) is WPO.

Then:

(i) \( x \leq f(x) \Rightarrow x \leq f^\infty(x) \);

(ii) \( x \geq f(x) \Rightarrow x \geq f^\infty(x) \).

**Abstract Comparison Lemma.** Let \((X, \rightarrow, \leq)\) be an ordered \(L\)-space and \(f, g, h : X \rightarrow X\) be such that:

(1) \( f \leq g \leq h \);

(2) the operators \(f, g, h\) are WPO;

(3) the operator \(g\) is increasing.

Then:

\[
x \leq y \leq z \Rightarrow f^\infty(x) \leq g^\infty(y) \leq h^\infty(z).
\]

Regarding the theory of WPO and PO see [18], [19], [22], [23], [24], [17], [21], [2], ...

### 2.3. Fiber Contraction Principle

In order to present our results, we need the following theorems (see [22], [25], [26], [27], ...).

**Fiber Contraction Theorem.** Let \((X, \rightarrow)\) be an \(L\)-space, \((Y, \rho)\) be a metric space, \(g : X \rightarrow X\), \(h : X \times Y \rightarrow Y\) and \(f : X \times Y \rightarrow X \times Y\), \(f(x, y) := (g(x), h(x, y))\). We suppose that:

(1) \((Y, \rho)\) is a complete metric space;

(2) \(g\) is WPO;

(3) \(h(x, \cdot) : Y \rightarrow Y\) is \(l\)-contraction, \(\forall x \in X\);

(4) \(h : X \times Y \rightarrow Y\) is continuous.

Then, \(f\) is WPO. Moreover, if \(g\) is a PO, then \(f\) is a PO.

**Generalized Fiber Contraction Theorem.** Let \((X, \rightarrow)\) be an \(L\)-space, \((X_i, d_i), i = \overline{1, m}, m \geq 1\) be metric spaces. Let, \(f_i : X_0 \times \ldots \times X_i \rightarrow X_i, i = \overline{0, m}\), be some operators. We suppose that:

(1) \((X_i, d_i), i = \overline{1, m}\), are complete metric spaces;

(2) \(f_0\) is a WPO;

(3) \(f_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \rightarrow X_i, i = \overline{1, m}\), are \(l_i\)-contractions;

(4) \(f_i, i = \overline{1, m}\), are continuous.

Then, the operator \(f : X_0 \times \ldots \times X_m \rightarrow X_0 \times \ldots \times X_m\), defined by,

\[
f(x_0, \ldots, x_m) := (f_0(x_0), f_1(x_0, x_1), \ldots, f_m(x_0, \ldots, x_m))
\]

is a WPO.

If \(f_0\) is a PO, then \(f\) is a PO.
3. Operators with Volterra property with respect to a subinterval

Let \((\mathbb{B}, +, \mathbb{R}, |.|)\) be a Banach space, \(a, b, c \in \mathbb{R}, a < c < b\). In what follows, we consider on \(C([a, b], \mathbb{B})\), \(C([a, c], \mathbb{B})\) norms of uniform convergence (max-norm, \(\|\|\), Bielecki norm, \(\|\|_\tau\)). In, \(C([a, b], \mathbb{B}) \times C([a, b], \mathbb{B})\), we consider a subset defined by,

\[
G := \{(x, y) \mid x, y \in C([a, b], \mathbb{B}), x\big|_{[a,c]} = y\big|_{[a,c]}\},
\]

and in, \(C([a, b], \mathbb{B})\), for each \(x \in C([a, c], \mathbb{B})\) we consider the subset,

\[
X_x := \{y \in C([a, b], \mathbb{B}) \mid y\big|_{[a,c]} = x\}.
\]

**Definition 3.1.** An operator, \(V : C([a, b], \mathbb{B}) \rightarrow C([a, b], \mathbb{B})\), has the Volterra property with respect to the subinterval, \([a, c]\), if the following implication holds,

\[
x, y \in C([a, b], \mathbb{B}), \ x\big|_{[a,c]} = y\big|_{[a,c]} \Rightarrow V(x)\big|_{[a,c]} = V(y)\big|_{[a,c]}.
\]

**Definition 3.2.** An operator, \(V : C([a, b], \mathbb{B}) \rightarrow C([a, b], \mathbb{B})\), has the Volterra property if it has the Volterra property with respect to each subinterval, \([a, t]\), for \(a < t < b\).

For example, let \(K, H \in C([a, b] \times [a, b] \times \mathbb{B}, \mathbb{B})\) and \(V : C([a, b], \mathbb{B}) \rightarrow C([a, b], \mathbb{B})\) be defined by,

\[
V(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds, \ t \in [a, b].
\]

This operator has the Volterra property with respect to the subinterval \([a, c]\), but \(V\) has not the Volterra property.

If, \(V : C([a, b], \mathbb{B}) \rightarrow C([a, b], \mathbb{B})\), is an operator with Volterra property with respect to \([a, c]\), then the operator \(V\) induces an operator, \(V_1\), on \(C([a, c], \mathbb{B})\), defined by

\[
V_1(x) := \tilde{V}(\tilde{x})\big|_{[a,c]}, \text{ where } \tilde{x} \in C([a, b], \mathbb{B}) \text{ with, } \tilde{x}\big|_{[a,c]} = x.
\]

**Remark 3.3.** If \(V\) has the Volterra property with respect to \([a, c]\) and \(V\) is a \(G\)-contraction (see section 2.1.), then the operator

\[
V\big|_{X_x} : X_x \rightarrow X_{V_1(x)},
\]

is a contraction for all \(x \in C([a, c], \mathbb{B})\). If \(x^* \in FV_1\), then, \(V(X_{x^*}) \subset X_{x^*}\).

The first abstract result of our paper is the following.

**Theorem 3.4.** In terms of the above notations, we suppose that:

1. \(V\) has the Volterra property with respect to \([a, c]\);
2. \(V_1\) is a contraction;
3. \(V\) is a \(G\)-contraction.

Then:

(i) \(F_V = \{x^*\}\);
(ii) \(x^*\big|_{[a,c]} = V_1^\infty(x), \ \forall \ x \in C([a, c], \mathbb{B})\);
(iii) \(x^* = V^\infty(x), \ \forall \ x \in X_{x^*}\big|_{[a,c]}\).
Proof. From (1) we have that, $F_{V_1} = \{ x^*_1 \}$, $x^*_1 \in C([a, c], \mathbb{B})$. From (3) and Remark 3.3, $V|_{X_{x^*_1}} : X_{x^*_1} \to X_{x^*_1}$, is a contraction, i.e., it has a unique fixed point, $x^*$, and $x^*|_{[a, c]} = x^*_1$. From these we have (i), (ii) and (iii).

**Conjecture 3.5.** In the conditions of Theorem 3.4, the operator $V$ is PO, i.e., $x^* = V^\infty(x)$, $\forall x \in C([a, b], \mathbb{B})$.

For a better understanding of Theorem 3.4 and Conjecture 3.5 in what follows, we present some examples.

**Example 3.6.** Let $a, b, c$ be as above and $\mathbb{B} := \mathbb{R}$. For $K, H \in C([a, b] \times [a, b] \times \mathbb{R})$ we consider the following functional integral equation,

$$x(t) = \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, \max_{\theta \in [a, s]} x(\theta))ds, \quad t \in [a, b]. \quad (3.1)$$

We are looking for the solution of this equation in $C[a, b]$. In addition, we suppose that:

(2') there exists $L_K > 0$ such that:

$$|K(t, s, u) - K(t, s, v)| \leq L_K|u - v|, \quad \forall t \in [a, b], \ \forall s \in [a, c], \ \forall u, v \in \mathbb{R};$$

(3') there exists $L_H > 0$ such that,

$$|H(t, s, u) - H(t, s, v)| \leq L_H|u - v|, \quad \forall t, s \in [a, b], \ \forall u, v \in \mathbb{R}.$$

In this case:

$$V(x)(t) = \text{the second part of (3.1)}; \quad V_1(x)(t) = \text{the second part of (3.1)}, \quad \forall t \in [a, c].$$

It is clear that $V$ has the Volterra property with respect to the subinterval $[a, c]$. We consider on $C[a, c]$ and $C[a, b]$ max-norms and if, $(L_K + L_H)(c - a) < 1$, the operator $V_1$ is a contraction and if, $L_H(b - c) < 1$, the operator $V$ is a $G$-contraction.

So, by Theorem 3.4, in the above conditions, equation (3.1) has in $C[a, b]$ a unique solution, $x^*$. Moreover, for $t \in [a, c]$, $x^*(t) = \lim_{n \to \infty} x_n(t)$, for each $x_0 \in C[a, c]$, where $\{x_n\}_{n \in \mathbb{N}}$ is defined by,

$$x_{n+1}(t) = \int_a^c K(t, s, x_n(s))ds + \int_a^t H(t, s, \max_{\theta \in [a, s]} x_n(\theta))ds,$$

and for $t \in [a, b]$, $x^*(t) = \lim_{n \to \infty} y_n(t)$, where $\{y_n\}_{n \in \mathbb{N}}$, is defined by

$y_0 \in C[a, b]$, with $y_0|_{[a, c]} = x^*|_{[a, c]}$, and

$$y_{n+1}(t) = \int_a^c K(t, s, x^*(s))ds + \int_a^t H(t, s, \max_{\theta \in [a, s]} y_n(\theta))ds.$$

**Remark 3.7.** In the case of operator $V$, in this example, Conjecture 3.5 is a theorem. Indeed, let $X_0 := C[a, c]$, $X_1 := C[c, b]$ and $C[a, b]$ be endowed with max-norms. We take, $f_0 := V_1$ and $f_1(x, y) : C[a, c] \times C[c, b] \to C[c, b]$ be defined by

$$f_1(x, y)(t) := \int_a^c K(t, s, x(s))ds + \int_a^c H(t, s, \max_{\theta \in [a, s]} x(\theta))ds + \int_a^c H(t, s, \max_{\theta \in [c, s]} y(\theta))ds.$$
We remark that, \( f_0 \) is a PO, and \( f_1(x, \cdot) : C[c, b] \to C[c, b] \) is \( L_H(b - c) \)-contraction. By Fiber Contraction Theorem, in the conditions, \((L_K + L_H)(c - a) < 1 \) and \( L_H(b - c) < 1 \), the operator \( f \) is a Picard operator. Let,

\[
x_0 \in C[a, c], \quad x_{n+1} = f_0(x_n), \quad n \in \mathbb{N},
\]
and

\[
y_0 \in C[c, b], \quad y_{n+1} = f_1(x_n, y_n), \quad n \in \mathbb{N}.
\]

Then, \( x_n \to x^*|_{[a, c]} \) as \( n \to \infty \), \( y_n \to x^*|_{[c, b]} \) as \( n \to \infty \).

We denote,

\[
u_n(t) = \begin{cases} x_n(t), & t \in [a, c], \\ y_n(t), & t \in [c, b]. \end{cases}
\]

Then, \( u_n \in C[a, b] \), for \( n \in \mathbb{N} \), and, \( u_{n+1} = V(u_n) \) with \( u_n \to x^* \) as \( n \to \infty \), i.e., \( V \) is a PO.

This result is very important because we can apply for \( V \), the Abstract Gronwall Lemma. So we have:

**Theorem 3.8.** Let us consider the equation (3.1) in the following conditions: \((L_K + L_H)(c - a) < 1 \), \( L_H(b - c) < 1 \) and \( K(t, s, \cdot) \), \( H(t, s, \cdot) : \mathbb{R} \to \mathbb{R} \) are increasing functions, for all \( t, s \in [a, b] \). Let us denote by \( x^* \) the unique solution of (3.1). Then the following implications hold:

(i) \( x \in C[a, b] \), \( x(t) \leq \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, \max \theta \in [a, s] x(\theta))ds \), \( t \in [a, b] \), \( \Rightarrow \) \( x \leq x^* \);

(ii) \( x \in C[a, b] \), \( x(t) \geq \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, \max \theta \in [a, s] x(\theta))ds \), \( t \in [a, b] \), \( \Rightarrow \) \( x \geq x^* \).

Also, from the Abstract Comparison Lemma we have a comparison result for equation (3.1).

**Remark 3.9.** For the functional integral equations with maxima, see [7], [17], [26], [22], [13], ...
3.4 In these conditions, equation (3.2) has in $C([a,b], \mathbb{B})$ a unique solution, $x^*$. Moreover, for $t \in [a,c]$, $x^*(t) = \lim_{n \to \infty} x_n(t)$, where $x_0 \in C[a,c]$, $x_{n+1}(t) = \int_a^c K(t,s,x_n(s))ds + \int_a^t H(t,s,x_n(s))ds$, $n \in \mathbb{N}$ and for $t \in [a,b]$, $x^*(t) = \lim_{n \to \infty} y_n(t)$, where $y_0 \in C([a,b], \mathbb{B})$, with $y_0|_{[a,c]} = x^*$, and $y_{n+1}(t) = \int_a^c K(t,s,x^*(s))ds + \int_a^t H(t,s,y_n(s))ds$, $n \in \mathbb{N}$.

Remark 3.11. In a similar way, as in the case of Example 3.6, the Conjecture 3.5 is a theorem for the operator $V$ in Example 3.10.

Remark 3.12. We can work, in the case of Example 3.10 with max-norm on $C([a,c], \mathbb{B})$ and with a Bielecki norm on $C[c,b]$, i.e., on $C([a,b], \mathbb{B})$ with the norm, $\|x\| = \max_{t \in [a,c]} \max_{s \in [a,b]} e^{-r(t-c)}|x(t)|$.

If $\mathbb{B} := \mathbb{R}^m$, then we can work with vectorial max-norms and with vectorial Bielecki norms.

Remark 3.13. For example of integral operator like $V$ in Example 3.10, which appear in differential equations, see: [3], [17], [4], [3] and the references in [3].

4. Operators with Volterra property

Let, $V : C([a,b], \mathbb{B}) \to C([a,b], \mathbb{B})$, be an operator with Volterra property. Let $m \in \mathbb{N}$, $m \geq 2$, $t_0 := a$, $t_1 := t_0 + \frac{b-a}{m}$, $\ldots$, $t_k := t_0 + \frac{k(b-a)}{m}$, $\ldots$, $t_m := b$. We denote by $V_k : C([t_0,t_k], \mathbb{B}) \to C([t_0,t_k], \mathbb{B})$, $k = 1, m-1$, the operators induced by $V$ on $[t_0, t_k]$ (see the definition of $V_1$ in section 3). We also consider the following sets, $G_k := \{(x,y) \mid x, y \in C([t_0, t_{k+1}], \mathbb{B}), x|_{[t_0,t_k]} = y|_{[t_0,t_k]}\}$, $k = 1, m-1$.

For, $x_k \in C([t_0, t_k], \mathbb{B})$, $k = 1, m-1$, we denote, $X_{x_k} := \{y \in C([t_0, t_k+1], \mathbb{B}) \mid y|_{[t_0,t_k]} = x_k\}$.

The second basic result of this paper is the following.

Theorem 4.1 (Theorem of step by step contraction). We suppose that:

1. $V$ has the Volterra property;
2. $V_1$ is a contraction;
3. $V_k$ is a $G_{k-1}$-contraction, for $k = \frac{2}{m}$.

Then:

(i) $F_V = \{x^*\}$;

(ii) $x^*|_{[t_0,t_1]} = V_1^\infty(x)$, $\forall x \in C([t_0,t_1], \mathbb{B})$,

$x^*|_{[t_0,t_2]} = V_2^\infty(x)$, $\forall x \in X_{x^*}|_{[t_0,t_1]}$,

$\vdots$

$x^*|_{[t_0,t_{m-1}]} = V_{m-1}^\infty(x)$, $\forall x \in X_{x^*}|_{[t_0,t_{m-2}]}$. 

(iii) \( x^* = V^\infty(x), \forall x \in X\big|_{[t_0,t_{m-1}]} \). 

**Proof.** It follows from successive (step by step !) application of Theorem 3.4 for the pairs, \((V_{k+1}, V_k)\), \(k = 1, m - 1\), with \(V_{k+1}\) as \(V\) and \(V_k\) as \(V_1\). 

**Conjecture 4.2.** In the condition of Theorem 4.1, the operator \(V\) is PO, with respect to uniform convergence on \(C([a,b], \mathbb{R})\).

**Example 4.3.** For \(K \in C([a,b] \times [a,b] \times \mathbb{R})\) we consider the following functional integral equation with maxima, 

\[
x(t) = \int_a^t K(t,s, \max_{\theta \in [a,s]} x(\theta))ds, \quad t \in [a,b] \tag{4.1}
\]

By step by step contraction principle we shall prove that, if there exists \(L_K > 0\) such that, 

\[
|K(t,s,u) - K(t,s,v)| \leq L_K|u - v|, \quad \forall t, s \in [a,b], \forall u, v \in \mathbb{R},
\]

then the equation (4.1) has in \([a,b]\) a unique solution. 

Indeed, let \(m \in \mathbb{N}^*\) be such that, \(\frac{L_K(b-a)}{m} < 1\). Let, \(V : C[a,b] \rightarrow C[a,b]\) be defined by, 

\[V(x)(t) := \text{the second part of (4.1)}.\]

First, we remark that \(V\) has the Volterra property. In this case:

\[V_1 : C[t_0,t_1] \rightarrow C[t_0,t_1], \quad V_1(x)(t) = \int_{t_0}^t K(t,s, \max_{\theta \in [t_0,s]} x(\theta))ds, \quad t \in [t_0,t_1].\]

A Lipschitz constant for \(V_1\) is, \(\frac{L_K(b-a)}{m}\). So, \(V_1\) is a contraction with respect to max-norm. 

In a similar way, \(V_2\) is a \(G_1\)-contraction, \(V_k\) is a \(G_{k-1}\)-contraction and \(V\) is \(G_{m-1}\)-contraction. So, we are in the conditions of Theorem 4.1. From this theorem we have that: The equation (4.1) has in \([a,b]\) a unique solution, \(x^*\). Moreover,

- for \(t \in [t_0,t_1]\), \(x^*(t) = \lim_{n \rightarrow \infty} x_n(t)\), where \(x_0 \in C[t_0,t_1]\), \(x_{n+1}(t) = \int_{t_0}^t K(t,s, \max_{\theta \in [t_0,s]} x_n(\theta))ds;\)
- for \(t \in [t_0,t_2]\), \(x^*(t) = \lim_{n \rightarrow \infty} x_n(t)\), where \(x_0 \in C[t_0,t_2]\) with \(x_0 |_{[t_0,t_1]} = x^* |_{[t_0,t_1]}\), and \(x_{n+1}(t) = \int_{t_0}^t K(t,s, \max_{\theta \in [t_0,s]} x_n(\theta))ds, \quad n \in \mathbb{N};\)
- for \(t \in [t_0,t_m]\), \(x^*(t) = \lim_{n \rightarrow \infty} x_n(t)\), where \(x_0 \in C[t_0,t_m]\) with \(x_0 |_{[t_0,t_{m-1}]} = x^* |_{[t_0,t_{m-1}]},\) and \(x_{n+1}(t) = \int_{t_0}^t K(t,s, \max_{\theta \in [t_0,s]} x_n(\theta))ds.\)

**Remark 4.4.** In a similar way as in the Example 3.6, by Generalized fiber contraction theorem, we have that, for \(V\) in Example 4.3, the Conjecture 4.2 is a theorem.

**Example 4.5.** For \(f \in C([a,b] \times \mathbb{R})\), we consider the following Cauchy problem

\[
x'(t) = f(t, \max_{\theta \in [a,t]} x(\theta)), \quad t \in [a,b] \tag{4.2}
\]

\[x(a) = 0 \tag{4.3}\]

This problem with \(x \in C^1[a,b]\) is equivalent with the following functional integral equation with maxima, in \(C[a,b]\),

\[x(t) = \int_a^t f(s, \max_{\theta \in [a,s]} x(\theta))ds.\]
From the result, in Example 4.3, we have that, if there exists \( L_f > 0 \) such that,
\[
|f(t,u) - f(t,v)| \leq L_f |u - v|, \quad \forall \ t \in [a, b], \ \forall \ u, v \in \mathbb{R},
\]
then the equation (4.3) has in \( C[a, b] \) a unique solution, i.e., the Cauchy problem (4.2) has in \( C^1[a, b] \) a unique solution.

**Remark 4.6.** For functional differential equations see: [1], [6], [11], [12], [16], [22], ...

**Remark 4.7.** For operators with Volterra property see: [10], [21], [15] and the references therein.

5. **Step by step generalized contraction principles**

There is a large class of generalized contraction principle (see, for example, [24], [2], [17]). As an example in what follows, we consider the case of \( \varphi \)-contractions.

Let \((X,d)\) be a metric space, \( G \subset X \times X \) a nonempty subset and \( f : X \to X \) be an operator.

**Definition 5.1.** Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a comparison function. By definition, \( f \) is a \((G,\varphi)\)-contraction if,
\[
d(f(x), f(y)) \leq \varphi(d(x,y)), \quad \forall \ x, y \in G.
\]

In the terms of notations in section 4, in a similar way as in the case of Theorem 4.1 we have:

**Theorem 5.2** (Theorem of step by step \( \varphi \)-contraction). We suppose that:

(1) \( V \) has the Volterra property;

(2) \( V_1 \) is a \( \varphi \)-contraction;

(3) \( V_k \) is a \((G_{k-1},\varphi)\)-contraction, for \( k = 2, m \).

Then:

(i) \( F_V = \{x^*\} \);

(ii)
\[
x^*|_{[t_0,t_1]} = V^\infty_1(x), \quad \forall \ x \in C([t_0,t_1], \mathbb{B}),
\]
\[
x^*|_{[t_0,t_2]} = V^\infty_2(x), \quad \forall \ x \in X_{x^*}|_{[t_0,t_1]},
\]
\[
\vdots
\]
\[
x^*|_{[t_0,t_{m-1}]} = V^\infty_{m-1}(x), \quad \forall \ x \in X_{x^*}|_{[t_0,t_{m-2}]}.
\]

(iii) \( x^* = V^\infty(x), \forall \ x \in X_{x^*}|_{[t_0,t_{m-1}]} \).

**Problem 5.3.** For which generalized contractions we have step by step corresponding result? If such generalized contractions are found, then the problem is to give relevant applications of such result.
References