Iterative approximation of common fixed points of generalized nonexpansive maps in convex metric spaces

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Abstract

We define SP-iteration procedure associated with three selfmaps $T_1, T_2, T_3$ defined on a nonempty convex subset of a convex metric space $X$ and prove $\Delta$-convergence of this iteration procedure to a common fixed point of $T_1, T_2, T_3$ under the hypotheses that each $T_i$ is either an $\alpha$-nonexpansive map or a Suzuki nonexpansive map in the setting of uniformly convex metric spaces. Also, we prove the strong convergence of this iteration procedure to a common fixed point of $T_1, T_2, T_3$ under certain additional hypotheses namely either semi-compact or condition (D).

Keywords: SP-iteration procedure, $\alpha$-nonexpansive map, Suzuki nonexpansive map, common fixed point, $\Delta$-convergence, strong convergence, uniformly convex metric space. 

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1. Introduction

In 1965, Browder [4] and Göhde [13] proved that every nonexpansive selfmap of a nonempty closed convex and bounded subset of a uniformly convex Banach space has a fixed point. Browder and Petryshyn [5, 6],
Senter and Dotson [19] used iteration procedures to approximate fixed points of nonexpansive maps in the setting of Banach spaces.

In 1970, Takahashi [22] introduced the concept of convexity in metric spaces as follows.

**Definition 1.1.** Let \((X, d)\) be a metric space. A map \(W : X \times X \times [0, 1] \to X\) is said to be a ‘convex structure’ on \(X\) if

\[
d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)
\]

for \(x, y, u \in X\) and \(\lambda \in [0, 1]\).

By a convex metric space, we mean a metric space \((X, d)\) together with a convex structure \(W\) and we denote it by \((X, d, W)\).

**Remark 1.2.** Every normed linear space \((X, ||.||)\) is a convex metric space. But there are convex metric spaces which are not normed linear spaces [3, 16, 22].

A nonempty subset \(K\) of \(X\) is said to be ‘convex’ if \(W(x, y, \lambda) \in K\) for \(x, y \in K\) and \(\lambda \in [0, 1]\).

Das and Debata [4] studied the convergence of common fixed points of a pair of quasi nonexpansive maps \(T_1, T_2\) by using the following iteration procedure in the setting of Banach spaces under certain hypotheses. Let \(X\) be a Banach space, \(K\) a nonempty convex subset of \(X\), \(T_1, T_2 : K \to K\) be selfmaps of \(K\). For \(x_1 \in K\),

\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1(\beta_n T_2 x_n) + (1 - \beta_n) x_n
\]

where \(\alpha_n, \beta_n \in [0, 1]\) for \(n \in \mathbb{N}\), where \(\mathbb{N}\) denote the set of all natural numbers.

Shimizu and Takahashi [20] introduced the notion of uniform convexity in convex metric spaces as follows.

**Definition 1.3.** [20] A convex metric space \((X, d, W)\) is said to be uniformly convex if for any \(\epsilon > 0\), there exists \(\alpha = \alpha(\epsilon)\) such that for all \(r > 0\) and \(x, y, z \in X\) with \(d(z, x) \leq r\), \(d(z, y) \leq r\) and \(d(x, y) \geq \epsilon r\),

\[
d(z, W(x, y, \frac{1}{2})) \leq r(1 - \alpha) < r.
\]

Takahashi and Tamura [23] studied the weak convergence of the iteration procedure [2] when both \(T_1\) and \(T_2\) are nonexpansive maps in the setting of Banach spaces, provided \(0 < a \leq \alpha_n, \beta_n \leq b < 1\) for \(n \in \mathbb{N}\).

Let \(T : K \to K\) be a map and \(K\), a nonempty subset of a metric space \((X, d)\). We denote \(F(T) = \{x \in K : Tx = x\}\), the set of all fixed points of \(T\).

A map \(T : K \to K\) is said to be a quasi nonexpansive map if \(F(T) \neq \emptyset\) and \(d(Tx, p) \leq d(x, p)\) for all \(x \in K\) and \(p \in F(T)\).

Suzuki [21] introduced a map with condition (C) in Banach spaces and under metric space setting it is as follows. Let \(K\) be a nonempty subset of a metric space \((X, d)\). A map \(T : K \to K\) is said to satisfy condition (C) if

\[
\frac{1}{2} d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in K.
\]

We call a map that satisfies condition (C), a Suzuki nonexpansive map. Aoyama and Kohsaka [11] introduced \(\alpha\)-nonexpansive maps in Banach spaces and under metric space setting it is as follows. Let \(K\) be a nonempty subset of a metric space \(X\). A map \(T : K \to K\) is said to be an \(\alpha\)-nonexpansive map for some \(\alpha < 1\) if

\[
d(Tx, Ty)^2 \leq \alpha d(Tx, x)^2 + \alpha d(Ty, y)^2 + (1 - 2\alpha) d(x, y)^2 \text{ for } x, y \in K.
\]

Aoyama and Kohsaka [11] observed the following facts:

(i) 0-nonexpansive map is called a nonexpansive map,
(ii) \(\frac{1}{2}\)-nonexpansive map is called a nonspreading map,
(iii) \(\frac{1}{3}\)-nonexpansive map is called a hybrid map.
For any bounded sequence \( \{x_n\} \) in a metric space \((X, d)\), the asymptotic radius with respect to \( K \subseteq X \) is defined by 
\[
\rho_K(\{x_n\}) = \inf_{x \in K} \{r(x, \{x_n\})\}
\]
where 
\[
r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n)
\]
and the asymptotic center of \( \{x_n\} \) with respect to \( K \) is defined by 
\[
A_K(\{x_n\}) = \{y \in K : r(y, \{x_n\}) = \rho_K(\{x_n\})\}.
\]
A sequence \( \{x_n\} \) in a metric space \((X, d)\) is said to \( \Delta \)-converges to a point \( x \in X \) if \( x \) is the unique asymptotic center for every subsequence \( \{x_{n_k}\} \) of the sequence \( \{x_n\} \). In this case, we write \( \Delta - \lim_{n \to \infty} x_n = x \).

A map \( T : K \to K \) is said to be semi-compact if every bounded sequence \( \{x_n\} \) in \( K \) with \( \lim_{n \to \infty} d(x_n, Tx_n) = 0 \) has a convergent subsequence.

Dhompongsa, Inthakon and Takahashi [8] proved that the sequence \( \{x_n\} \) generated by the iteration procedure (2) converges weakly to a common fixed point of \( T_1 \) and \( T_2 \), where \( T_1 \) is a nonspreading map and \( T_2 \) is a Suzuki nonexpansive map in the setting of Hilbert spaces, provided 
\[
\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0 \quad \text{and} \quad \liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0.
\]

In 2011, Phuengrattana and Suantai [13] introduced a three step iteration procedure namely SP-iteration procedure to approximate fixed points of a continuous nondecreasing function defined on a closed interval on the real line and proved that this iteration procedure converges faster than Mann iteration procedure [15], Ishikawa iteration procedure [14] and Noor iteration procedure [17]. In the setting of normed linear spaces, SP-iteration procedure is defined as follows.

Let \( K \) be a nonempty convex subset of a normed linear space \( X \), 
\( T : K \to K \) be a selfmap of \( K \) and for any \( x_0 \in K \),
\[
z_n = (1 - \gamma_n)x_n + \gamma_nTx_n
\]
\[
y_n = (1 - \beta_n)z_n + \beta_nTz_n
\]
\[
x_{n+1} = (1 - \alpha_n)y_n + \alpha_nTy_n
\]
where \( \{\alpha_n\}_{n=0}^\infty \), \( \{\beta_n\}_{n=0}^\infty \) and \( \{\gamma_n\}_{n=0}^\infty \) are sequences in \([0, 1]\).

In 2013, Wattanawitoon and Khanlale [24] considered the iteration procedure (2) to approximate common fixed points of \( T_1 \) and \( T_2 \), where \( T_1 \) is an \( \alpha \)-nonexpansive map and \( T_2 \) is a Suzuki nonexpansive map in the setting of Hilbert spaces, provided \( 0 < a \leq \alpha_n, \beta_n \leq b < 1 \).

Uddin and Imdad [24] studied \( \Delta \)-convergence and strong convergence of SP-iteration procedure to compute fixed points of Suzuki nonexpansive mappings in Hadamard spaces.

Recently, Hafiz Fukhar-ud-din [11] considered one step iteration procedure and proved the following convergence theorem in the setting of convex metric spaces.

**Theorem 1.5.** Let \( K \) be a nonempty, closed and convex subset of a complete and uniformly convex metric space \( X \) with continuous convex structure \( W \). Let \( T \) be an \( \alpha \)-nonexpansive selfmap on \( K \), \( S \) a selfmap of \( K \) satisfying condition (C). For \( x_1 \in K \), define
\[
x_{n+1} = W(Tx_n, W(Sx_n, x_n, \frac{\beta_n}{1 - \alpha_n}), \alpha_n)
\]
where \( 0 < a \leq \alpha_n, \beta_n \leq b < 1 \) for \( n \in \mathbb{N} \).
Let \( F \) be the set of all common fixed points of \( S \) and \( T \). If \( F \neq \emptyset \) then \( \Delta - \lim_{n \to \infty} x_n = x \in F \). Moreover, if either \( S \) or \( T \) is semi-compact then \( \{x_n\} \) converges strongly to a point of \( F \).

For more literature on this topic, we refer to [10] [12] and related references there in.

The following lemmas are useful in developing this paper.

**Lemma 1.6.** [21] Let \( T \) be a selfmap defined on a nonempty subset \( K \) of a metric space \((X, d)\). If \( T \) satisfies condition (C) then 
\[
d(x, Ty) \leq 3d(Tx, x) + d(x, y)
\]
for all \( x, y \in K \).
Lemma 1.7. [12] Let $K$ be a nonempty, closed and convex subset of a metric space $X$ and $T$ be an $\alpha$-nonexpansive mapping on $K$. For any $x, y \in K$, the following two assertions hold:

(i) If $0 \leq \alpha < 1$ then $d(x, Ty)^2 \leq \frac{1+\alpha}{1-\alpha}d(x, Tx)^2 + \frac{2}{1-\alpha}\{\alpha d(x, y) + d(Tx, Ty)\}d(x, Tx) + d(x, y)^2$.

(ii) If $\alpha < 0$ then $d(x, Ty)^2 \leq d(x, Tx)^2 + \frac{2}{1-\alpha}\{d(Tx, Ty) - \alpha d(Tx, y)\}d(x, Tx) + d(x, y)^2$.

A sequence $\{x_n\}_{n=0}^{\infty}$ in a metric space $(X, d)$ is said to be a Fejér monotone sequence with respect to a subset $C$ of $X$ if $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in C$ and $n \in \mathbb{N} \cup \{0\}$.

For any subset $A$ of a metric space $(X, d)$ and $x \in X$, we denote $\text{dist}(x, A) = \inf_{y \in A}\{d(x, y)\}$.

Lemma 1.8. [2] Let $K$ be a nonempty closed subset of a complete metric space $(X, d)$ and $\{x_n\}$ a Fejér monotone sequence with respect to $K$. Then $\{x_n\}$ converges to some point $x \in K$ if and only if $\lim_{n \to \infty} \text{dist}(x_n, K) = 0$.

Lemma 1.9. [9] Let $K$ be a nonempty, closed and convex subset of a uniformly convex complete metric space $(X, d, W)$. Then every bounded sequence $\{x_n\}$ in $X$ has a unique asymptotic center with respect to $K$.

Lemma 1.10. [10] Let $X$ be a uniformly convex metric space with continuous convex structure $W$. Let $x \in X$ and $\{a_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{u_n\}$ and $\{v_n\}$ are sequences in $X$ such that $\limsup_{n \to \infty} d(u_n, x) \leq r$, $\limsup_{n \to \infty} d(v_n, x) \leq r$ and $\lim_{n \to \infty} d(W(u_n, v_n, a_n), x) = r$ for some $r \geq 0$, then $\lim_{n \to \infty} d(u_n, v_n) = 0$.

Motivated by the works of Das and Debata [7], Takahashi and Tamura [23], Dhompongsa, Inthakon and Takahashi [8], Wattanawitoon and Khamlae [24], Uddin and Imdad [24] and Hafiz Fukhar-ud-din [11], in this paper, we define SP-iteration procedure associated with three selfmaps $T_1, T_2, T_3$ in convex metric spaces and prove the $\Delta$-convergence of this iteration procedure to a common fixed point of $T_1, T_2, T_3$ under the hypotheses that each $T_i$ is either an $\alpha$-nonexpansive map or a Suzuki nonexpansive map. Further, with an additional assumption that either any one of $T_1, T_2, T_3$ is semi-compact or $T_1, T_2, T_3$ satisfies condition (D), we prove the strong convergence of this iteration procedure to a common fixed point of $T_1, T_2$ and $T_3$.

2. Convergence of SP-iteration associated with three maps

Let $(X, d, W)$ be a convex metric space and $K$ be a nonempty convex subset of $X$. Let $T_1, T_2, T_3 : K \to K$ be three selfmaps of $K$. A point $x \in K$ is said to be a common fixed point of $T_1, T_2, T_3$ if $T_1x = T_2x = T_3x = x$. We denote the set of all common fixed points of $T_1, T_2$, and $T_3$ by $F = \bigcap_{i=1}^{3} F(T_i)$.

We define SP-iteration procedure associated with three selfmaps in the setting of convex metric spaces as follows. Let $K$ be a nonempty convex subset of a convex metric space $X$, and $T_1, T_2, T_3 : K \to K$ be three selfmaps. For $x_0 \in K$,

$$
\begin{align*}
    z_n &= W(T_1x_n, x_n, \gamma_n) \\
    y_n &= W(T_2z_n, z_n, \beta_n) \\
    x_{n+1} &= W(T_3y_n, y_n, \alpha_n)
\end{align*}
$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ for $n \in \mathbb{N} \cup \{0\}$.

Lemma 2.1. Let $K$ be a nonempty convex subset of a convex metric space $(X, d, W)$. Let $T_1, T_2, T_3 : K \to K$ be selfmaps of $K$ such that $F \neq \emptyset$. Assume that each $T_i$ is either an $\alpha$-nonexpansive or a Suzuki nonexpansive map. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by SP-iteration procedure associated with three selfmaps [7]. Then
(i) $\{x_n\}$ is a Fejér monotone sequence with respect to $F$,
(ii) $\lim_{n \to \infty} d(x_n, p)$ exists for each $p \in F$, and
(iii) $\lim_{n \to \infty} \text{dist}(x_n, F)$ exists.

Proof. Let $p \in F$ and $n \in \mathbb{N} \cup \{0\}$. We consider $d(x_{n+1}, p) = d(W(T_3y_n, y_n, \alpha_n), p) \leq \alpha_n d(T_3y_n, p) + (1 - \alpha_n)d(y_n, p) \leq \alpha_n d(y_n, p) + (1 - \alpha_n)d(y_n, p)$ (since $T_3$ is quasi nonexpansive) $= d(y_n, p)$. (8)

We now consider $d(y_n, p) = d(W(T_2z_n, z_n, \beta_n), p) \leq \beta_n d(T_2z_n, p) + (1 - \beta_n)d(z_n, p) \leq \beta_n d(z_n, p) + (1 - \beta_n)d(z_n, p)$ (since $T_2$ is quasi nonexpansive) $= d(z_n, p)$. (9)

Now, we consider $d(z_n, p) = d(W(T_1x_n, x_n, \gamma_n), p) \leq \gamma_n d(T_1x_n, p) + (1 - \gamma_n)d(x_n, p) \leq \gamma_n d(x_n, p) + (1 - \gamma_n)d(x_n, p)$ (since $T_1$ is quasi nonexpansive) $= d(x_n, p)$ for $n \in \mathbb{N} \cup \{0\}$. (10)

Therefore from the inequalities (8), (9) and (10), we have $d(x_{n+1}, p) \leq d(x_n, p)$ (11) for all $p \in F$ and $n \in \mathbb{N} \cup \{0\}$, that is, $\{x_n\}$ is a Fejér monotone sequence with respect to $F$.

We observe from the inequality (11) that the sequence $\{d(x_n, p)\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative real numbers so that $\lim_{n \to \infty} d(x_n, p)$ exists for each $p \in F$.

By using the inequality (11), it is easy to see that $\text{dist}(x_{n+1}, F) \leq d(x_n, p)$ for all $p \in F$ so that $\text{dist}(x_{n+1}, F) \leq \text{dist}(x_n, F)$ for $n \in \mathbb{N} \cup \{0\}$ and hence $\lim_{n \to \infty} \text{dist}(x_n, F)$ exists. \hfill $\Box$

Lemma 2.2. Let $K$ be a nonempty closed and convex subset of a uniformly convex metric space $X$ with continuous convex structure $W$. Let $T_1, T_2, T_3 : K \to K$ be selfmaps and assume that each $T_i$ is either an α-nonexpansive map or a Suzuki nonexpansive map. Assume that $F \neq \emptyset$. Let $\alpha_n, \beta_n, \gamma_n \in [0, 1)$ for $n \in \mathbb{N} \cup \{0\}$. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the SP-iteration procedure associated with three selfmaps (7). Then $\lim_{n \to \infty} d(x_n, T_i x_n) = 0$ for $i = 1, 2, 3$.

Proof. It follows from (ii) of Lemma 2.1 that for each $p \in F$, there exists $c \geq 0$ such that $\lim_{n \to \infty} d(x_n, p) = c$.

Therefore from the inequalities (8), (9) and (10), we have $d(x_{n+1}, p) \leq d(y_n, p) \leq d(z_n, p) \leq d(x_n, p)$ and hence $\lim_{n \to \infty} d(y_n, p) = c$, and $\lim_{n \to \infty} d(z_n, p) = c$.

Since $\lim_{n \to \infty} d(z_n, p) = c$, we have $\lim_{n \to \infty} d(W(T_1x_n, x_n, \gamma_n), p) = c$. Since $T_1$ is a quasi nonexpansive map, we have
By (i) of Lemma 1.7, we have
\[ \limsup_{n \to \infty} d(x_n, T_1 x_n) = 0. \] (12)

We consider
\[ d(z_n, x_n) = d(W(T_1 x_n, x_n, \gamma_n), x_n) \leq \gamma_n d(T_1 x_n, x_n) \leq b d(T_1 x_n, x_n). \]
On letting \( n \to \infty \), we have
\[ \lim_{n \to \infty} d(z_n, x_n) = 0. \] (13)

Since \( \lim_{n \to \infty} d(y_n, p) = c \), we have \( \lim_{n \to \infty} d(W(T_2 z_n, z_n, \beta_n), p) = c \).
Since \( T_2 \) is a quasi nonexpansive map, we have
\[ \limsup_{n \to \infty} d(T_2 z_n, p) \leq \limsup_{n \to \infty} d(z_n, p) = c. \]
Therefore by Lemma 1.10, we have
\[ \lim_{n \to \infty} d(z_n, T_2 z_n) = 0. \] (14)

We consider
\[ d(y_n, x_n) = d(W(T_2 z_n, z_n, \beta_n), x_n) \]
\[ \leq \beta_n d(T_2 z_n, x_n) + (1 - \beta_n)d(z_n, x_n) \]
\[ \leq \beta_n d(T_2 z_n, z_n) + d(z_n, x_n) \text{ for } n \in \mathbb{N} \cup \{0\}. \]
Now, on letting \( n \to \infty \) it follows from (13) and (14) that
\[ \lim_{n \to \infty} d(y_n, x_n) = 0. \] (15)

Since \( \lim_{n \to \infty} d(x_{n+1}, p) = c \), we have \( \lim_{n \to \infty} d(W(T_3 y_n, y_n, \alpha_n), p) = c. \)
Since \( T_3 \) is a quasi nonexpansive map, we have
\[ \limsup_{n \to \infty} d(T_3 y_n, p) \leq \limsup_{n \to \infty} d(y_n, p) = c. \]
Again by Lemma 1.10, we have
\[ \lim_{n \to \infty} d(y_n, T_3 y_n) = 0. \] (16)

Now, we prove \( \lim_{n \to \infty} d(x_n, T_i x_n) = 0 \) for \( i = 2, 3 \) by considering the following cases.

**Case (i):** \( T_2 \) is a Suzuki nonexpansive map.
By using triangle inequality and Lemma 1.6, it is easy to see that
\[ d(x_n, T_2 x_n) \leq d(x_n, z_n) + d(z_n, T_2 x_n) \]
\[ = 2d(z_n, x_n) + 3d(z_n, T_2 z_n) \text{ for } n \in \mathbb{N} \cup \{0\}. \]
Therefore it follows from (13) and (14) that \( \lim_{n \to \infty} d(x_n, T_2 x_n) = 0. \)

**Case (ii):** \( T_2 \) is an \( \alpha \)-nonexpansive map for \( 0 \leq \alpha < 1 \).
By (i) of Lemma 1.7, we have
\[ d(z_n, T_2 x_n)^2 \leq \frac{1}{1-\alpha} d(z_n, T_2 z_n)^2 + \frac{2}{1-\alpha} \{\alpha d(z_n, x_n) + A\} d(z_n, T_2 z_n) + d(z_n, x_n)^2 \]
where \( A = \sup\{d(T_2 z_n, T_2 x_n) : n \in \mathbb{N} \cup \{0\}\} \).
On letting \( n \to \infty \), it follows from (13) and (14) that \( \lim_{n \to \infty} d(z_n, T_2 x_n) = 0. \)
Now by using the triangle inequality, it follows that \( \lim_{n \to \infty} d(x_n, T_2 x_n) = 0. \)

**Case (iii):** \( T_2 \) is an \( \alpha \)-nonexpansive map for \( \alpha < 0 \).
By (ii) of Lemma 1.7, we have
\[ d(z_n, T_2 x_n)^2 \leq d(z_n, T_2 z_n)^2 + \frac{1}{1-\alpha} \{d(T_2 z_n, T_2 x_n) - \alpha d(T_2 z_n, z_n)\} d(z_n, T_2 z_n) \]
\[ + d(z_n, x_n)^2 \]
\[ \leq d(z_n, T_2 z_n)^2 + \frac{2}{1-\alpha} \{A - \alpha d(T_2 z_n, z_n) - \alpha d(z_n, x_n)\} d(z_n, T_2 z_n) \]
\[ + d(z_n, x_n)^2 \]
where \( A \) is defined as in case (ii).
On letting $n \to \infty$, it is easy to see from (13) and (14) that
\[
\lim_{n \to \infty} d(x_n, T_2 x_n) = 0 \quad \text{and hence} \quad \lim_{n \to \infty} d(x_n, T_2 x_n) = 0.
\]

Case (iv): $T_3$ is either a Suzuki nonexpansive map or an $\alpha$-nonexpansive map for $\alpha < 1$.

By proceeding as in the above cases, it follows from (15) and (16) that
\[
\lim_{n \to \infty} d(x_n, T_3 x_n) = 0.
\]

**Theorem 2.3.** Let $K$ be a nonempty, closed and convex subset of a complete and uniformly convex metric space $X$ with continuous convex structure $W$. Let $T_1, T_2, T_3 : K \to K$ be three selfmaps of $K$ such that each $T_i$ is either an $\alpha$-nonexpansive map or a Suzuki nonexpansive map. Assume that $F \neq \emptyset$ and let $\alpha_n, \beta_n, \gamma_n \in [a, b] \subseteq (0, 1)$ for $n \in \mathbb{N} \cup \{0\}$. For any $x_0 \in K$, let $\{x_n\}$ be the sequence generated by SP-iteration procedure associated with three selfmaps (7). Then there exists $x \in F$ such that $\Delta - \lim_{n \to \infty} x_n = x$.

**Proof.** By Lemma 2.1, we have $\{x_n\}$ is bounded. Therefore by Lemma 1.9, the sequence $\{x_n\}$ has a unique asymptotic center with respect to $K$, i. e., $A_K(\{x_n\}) = \{x\}$ for some $x \in K$. Similarly, if $\{x_n\}$ is a subsequence of the sequence $\{x_n\}$, then there exists $u \in K$ such that $A_K(\{x_n\}) = \{u\}$.

We substitute $x = x_n$ and $y = x$ in Lemma 1.6 if $T_i$ a Suzuki nonexpansive map, and in Lemma 1.7 if $T_i$ is an $\alpha$-nonexpansive map through which it follows that $\limsup_{n \to \infty} d(x_n, T_i x) \leq \limsup_{n \to \infty} d(x_n, x)$ so that $r_K(\{x_n\}) = \limsup_{n \to \infty} d(x_n, T_i x)$ for $i = 1, 2, 3$. Therefore $T_i x \in A_K(\{x_n\}) = \{x\}$ for $i = 1, 2, 3$ so that $x \in F$. Similarly, we have $u \in F$.

Now, we prove that $x = u$. On the contrary, let $x \neq u$. Since $u \in F$, it follows from Lemma 2.1 that $\lim_{n \to \infty} d(x_n, u)$ exists. Now, we consider
\[
\limsup_{k \to \infty} d(x_n, u) < \limsup_{k \to \infty} d(x_n, x) \quad \text{since} \quad A_K(\{x_n\}) = \{u\}
\leq \limsup_{k \to \infty} d(x_n, x) < \limsup_{n \to \infty} d(x_n, u) \quad \text{since} \quad A_K(\{x_n\}) = \{x\}
= \lim_{n \to \infty} d(x_n, u) = \lim_{k \to \infty} d(x_n, u) = \limsup_{k \to \infty} d(x_n, u),
\]
a contradiction.

Therefore $A_K(\{x_n\}) = \{x\}$ for every subsequence $\{x_n\}$ of the sequence $\{x_n\}$, that is, $\Delta - \lim_{n \to \infty} x_n = x$.

**Theorem 2.4.** Under the hypotheses of Theorem 2.3, if any one of $T_1, T_2$, or $T_3$ is semi-compact then for $x_0 \in K$, the sequence $\{x_n\}$ generated by SP-iteration procedure (7) associated with three selfmaps converges strongly to a common fixed point of $T_1, T_2$, and $T_3$.

**Proof.** Let $T_i$ be semi-compact for some $i = 1, 2, 3$.

Now by Lemma 2.1 and Lemma 2.2, we have the sequence $\{x_n\}$ is bounded and $\lim_{n \to \infty} d(x_n, T_i x_n) = 0$. Since $T_i$ is semi-compact, the sequence $\{x_n\}$ has a subsequence $\{x_{nk}\}$ such that $\lim_{k \to \infty} d(x_{nk}, x) = 0$ for some $x \in K$.

Now we prove that $x \in F$.

Case (i): $T_j$ is a Suzuki nonexpansive map for $j = 1, 2, 3$.

By using Lemma 1.6, we have $d(x_{nk}, T_j x) \leq 3d(x_{nk}, T_j x) + d(x_{nk}, x)$ for all $k$ so that $\lim_{k \to \infty} d(x_{nk}, T_j x) = 0$ and hence $x \in F(T_j)$.

Case (ii): $T_j$ is an $\alpha$-nonexpansive map for $0 \leq \alpha < 1$ and $j = 1, 2, 3$.

By Lemma 1.7, we have
\[
d(x_{nk}, T_j x)^2 \leq \frac{1+\alpha}{1-\alpha} d(x_{nk}, T_j x) + \frac{\alpha}{1-\alpha} (d(x_{nk}, x) + d(T_j x, x)) + d(x_{nk}, T_j x) + d(x_{nk}, x)^2
\]
\[
\leq \frac{1+\alpha}{1-\alpha} d(x_{nk}, T_j x) + \frac{\alpha}{1-\alpha} (d(x_{nk}, x) + d(T_j x, x)) + d(x_{nk}, T_j x) + d(x_{nk}, x)^2.
\]

On letting $k \to \infty$, we have $\lim_{k \to \infty} d(x_{nk}, T_j x) = 0$ and hence $x \in F(T_j)$. 

Case (iii): \( T_j \) is an \( \alpha \)-nonexpansive map for \( \alpha < 0 \) and \( j = 1, 2, 3. \)
By proceeding as in case (ii), it follows from Lemma 1.7 that \( x \in F(T_j). \)
Hence by considering all the above cases, we have \( x \in F. \)

Therefore by Lemma 2.1 we have \( \lim_{n \to \infty} d(x_n, x) \) exists and hence the sequence \( \{x_n\} \) converges strongly to a point \( x \in F. \)

We say that three selfmaps \( T_1, T_2, T_3 : K \to K \) are said to satisfy condition (D) with respect to a subset \( C \) of \( K \) if there exists a nondecreasing function \( f : [0, \infty) \to [0, \infty) \) with \( f(0) = 0 \) and \( f(t) > 0 \) for all \( t > 0 \) such that

\[
    f(\text{dist}(x, C)) \leq \sum_{i=1}^{3} d(x, T_i x) \quad \text{for all } x \in K. \tag{17}
\]

**Theorem 2.5.** Under the hypotheses of Theorem 2.3 if \( T_1, T_2, \) and \( T_3 \) satisfy condition (D) with respect to \( F \) then the sequence \( \{x_n\}_{n=0}^{\infty} \) generated by SP-iteration procedure (7) associated with three selfmaps converges strongly to a common fixed point of \( T_1, T_2, \) and \( T_3. \)

**Proof.** By Lemma 2.2 we have \( \lim_{n \to \infty} d(x_n, T_i x_n) = 0 \) for \( i = 1, 2, 3 \) so that

\[
    \lim_{n \to \infty} \sum_{i=1}^{3} d(x_n, T_i x_n) = 0. \quad \text{Therefore from the inequality (17), we have}
\]

\[
    \lim_{n \to \infty} f(\text{dist}(x_n, F)) = 0.
\]

We now prove that \( \lim_{n \to \infty} \text{dist}(x_n, F) = 0. \) On the contrary, if \( \lim_{n \to \infty} \text{dist}(x_n, F) \neq 0 \) then there exist an \( \epsilon > 0 \) and a subsequence \( \{x_{n_k}\} \) of the sequence \( \{x_n\} \) such that

\[
    \text{dist}(x_{n_k}, F) \geq \epsilon \quad \text{for all } k.
\]

Therefore \( f(\text{dist}(x_{n_k}, F)) \geq f(\epsilon) > 0 \) for all \( k \) so that

\[
    \lim_{n \to \infty} f(\text{dist}(x_n, F)) \neq 0, \quad \text{a contradiction.}
\]

Therefore \( \lim_{n \to \infty} \text{dist}(x_n, F) = 0. \)

By Lemma 2.1 we have the sequence \( \{x_n\} \) is a Fejér monotone sequence with respect to \( F. \) Since \( T_1, T_2, \) and \( T_3 \) are quasi nonexpansive maps, we have \( F \) is closed. Therefore by using Lemma 1.8 we have the sequence \( \{x_n\} \) converges strongly to a point of \( F. \)

By choosing \( \alpha = \frac{1}{2} \) or \( \alpha = \frac{1}{3} \) in Theorem 2.3 Theorem 2.4 and Theorem 2.5 we have the following corollary.

**Corollary 2.6.** Let \( K \) be a nonempty, closed and convex subset of a complete and uniformly convex metric space \( X \) with continuous convex structure \( W. \) Let \( T_1, T_2, T_3 : K \to K \) be three selfmaps of \( K \) such that each \( T_i \) is a nonspreading map, a hybrid map, or a Suzuki nonexpansive map. Assume that \( F \neq \emptyset. \) Let \( \alpha_n, \beta_n, \gamma_n \in [a, b] \subseteq (0, 1) \) for \( n \in \mathbb{N} \cup \{0\}. \) For any \( x_0 \in K, \) let \( \{x_n\} \) be the sequence generated by SP-iteration procedure (7) associated with three selfmaps. Then

(a) there exists \( x \in F \) such that \( \Delta - \lim_n x_n = x, \)

(b) if any one of \( T_1, T_2, \) and \( T_3 \) is semi-compact then \( \{x_n\} \) converges strongly to a common fixed point of \( T_1, T_2, T_3, \) and

(c) if \( T_1, T_2, \) and \( T_3 \) satisfy condition (D) with respect to \( F \) then the sequence \( \{x_n\}_{n=0}^{\infty} \) converges strongly to a common fixed point of \( T_1, T_2, \) and \( T_3. \)

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References