Set Inner Amenability for Semigroups

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Abstract

In this paper, we present a new concept of inner amenability for a non-empty arbitrary subset $A$ of discrete semigroup $S$ called $A$-inner amenability. This condition is considerably weaker than ordinary inner amenability. Further, we show some relationships between this version of inner amenability and Følner’s condition.

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1. Introduction

Throughout this paper, $S$ will denote a discrete semigroup. We shall use $\ell^\infty(S)$ to denote the Banach space of bounded real-valued functions on $S$ with the supremum norm. For every subset $A$ of $S$, let $\chi_A$ denote its characteristic function, that is

$$
\chi_A(s) = \begin{cases} 
1 & s \in A \\
0 & s \notin A
\end{cases}
$$

A mean is a linear functional $m \in \ell^\infty(S)^*$ such that $m(\chi_S) = \|m\| = 1$. For each $s \in S$ and $f \in \ell^\infty(S)$ we define $sf$ and $fs$ on $S$ by $(sf)(t) = f(st)$ and $(fs)(t) = f(ts)$ for all $t \in S$. We say that $m \in \ell^\infty(S)^*$ is invariant if $m(sf) = m(f) = m(fs)$ for all $s \in S$ and $f \in \ell^\infty(S)$. A semigroup $S$ is said to be amenable if it has an invariant mean $m$ on $\ell^\infty(S)$. Also, let $\ell^1(S)$ denote the Banach space of all real-valued functions $\varphi$ on $S$ such that $\|\varphi\|_1 := \sum_{x \in S} |\varphi(x)| < \infty$. With pointwise addition and scalar multiplication, and with convolution

$$
(\varphi \ast \psi)(x) = \sum_{st=x} \varphi(s)\psi(t) \quad (x \in S),
$$

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as product, \( \ell^1(S) \) is a Banach algebra.

We say that \( m \in \ell^\infty(S)^* \) is inner invariant mean if
\[
m(a f) = m(f a),
\]
for all \( s \in S \) and \( f \in \ell^\infty(S) \). Following Ling [11], a semigroup \( S \) is said to be inner amenable if it has an inner invariant mean \( m \) on \( \ell^\infty(S) \).

We will show that many results concerning inner amenability of semigroups have similar analogues for \( A \)-inner amenability. Finally, a number of equivalent conditions characterizing \( A \)-inner amenable semigroups is given.

2. Set inner amenability for semigroup

We start off with the following definition, which is the most important here.

**Definition 2.1.** Let \( S \) be a semigroup and \( \phi \neq A \subseteq S \). We say that a mean \( m \) on \( \ell^\infty(S) \), is an inner \( A \)-invariant mean if for all \( a \in A \) and \( f \in \ell^\infty(S) \) we have
\[
m(a f) = m(f a).
\]

A semigroup \( S \) which admits inner \( A \)-invariant means is called \( A \)-inner amenable.

In other words, invariance of \( m \) is only required in the subsets of \( S \). It follows immediately that every inner amenable semigroup is \( A \)-inner amenable for all subsets \( A \) of \( S \). But the converse is not true in general. (see Examples 3.2 and 3.4)

For an arbitrary non-empty subset \( A \) of semigroup \( S \), we denote by \( \mathcal{H}(A) \), the real linear span of functions of the form \( a f - f a \), where \( a \in A \) and \( f \in \ell^\infty(S) \). In the following theorem, a sequence of characterizations of \( A \)-inner amenable semigroup is given.

**Theorem 2.2.** Let \( S \) be a semigroup with non-empty subset \( A \). Then the following properties are equivalent:

(a) \( S \) is an \( A \)-inner amenable semigroup.

(b) for every \( h \in \mathcal{H}(A) \), \( \sup\{h(x) : x \in S\} \geq 0 \).

(c) \( \inf\{\|1 - h\|_\infty : h \in \mathcal{H}(A)\} = 1 \).

**Proof.** (a) \(\Rightarrow\) (b). Let \( m \) be an inner \( A \)-invariant mean on \( \ell^\infty(S) \). If \( h \in \mathcal{H}(A) \), then \( \sup\{h(x) : x \in S\} \geq m(h) = 0 \). Thus, the property (b) holds.

(b) \(\Rightarrow\) (c). For every \( h \in \mathcal{H}(A) \), we have
\[
0 \leq \sup\{-h(x) : x \in S\} = - \inf\{h(x) : x \in S\}.
\]

This shows that, \( \inf\{h(x) : x \in S\} \leq 0 \). Hence, for any \( \epsilon > 0 \), there exists \( x_0 \in S \) such that \( h(x_0) < \epsilon \), and so \( 1 - h(x_0) > 1 - \epsilon \). Therefore, \( \|1 - h\|_\infty \geq 1 \) for any \( h \in \mathcal{H}(A) \). But \( 0 \in \mathcal{H}(A) \), \( \inf\{\|1 - h\|_\infty : h \in \mathcal{H}(A)\} \leq \|1 - 0\|_\infty = 1 \).

(c) \(\Rightarrow\) (a). Assume that the property (c) holds. Now by the Hahn-Banach theorem, there exists a linear functional \( m \) on \( \ell^\infty(S) \) with norm one such that \( m(\mathcal{H}(A)) = \{0\} \) and \( m(1) = \inf\{\|1 - h\|_\infty : h \in \mathcal{H}(A)\} \). So \( m \) is an inner \( A \)-invariant mean on \( \ell^\infty(S) \).

A non-empty subset \( A \) of \( S \) is said to act injectively on the left (right) of semigroup \( S \), if \( ax = ay \) (\( xa = ya \)) implies \( x = y \) for every \( a \in A, x, y \in S \). We say that \( A \) acts injectively on the semigroup \( S \), if it acts on both left and right of \( S \). In particular, if \( S \) is a cancellative semigroup, then every non-empty subset of \( S \) acts injectively on \( S \).
Theorem 2.3. Let $A$ act injectively on the left of semigroup $S$. Then $S$ is $A$-inner amenable if and only if $\mathcal{H}(A)$ is not norm dense in $\ell^\infty(S)$.

Proof. We suppose that $m$ be a nonzero self-adjoint functional $m \in \ell^\infty(S)^*$ such that $m(\mathcal{H}(A)) = 0$. Consider the decomposition $m = m^+ - m^-$, such that

$$m^+(f) = \sup\{m(g) : 0 \leq g \leq f\}$$

and

$$m^-(f) = -\inf\{m(g) : 0 \leq g \leq f\},$$

for all $f \in \ell^\infty(S)$ with $f \geq 0$. A similar proof of Theorem 2 of [11], shows that $m^+$ and $m^-$ are inner $A$-invariant mean on $\ell^\infty(S)$. \hfill \Box

In the following proposition, we see that increasing union of a family of $A$-inner amenable semigroups is $A$-inner amenable.

Proposition 2.4. Let $\{S_\alpha\}_{\alpha \in I}$ be a family of subsemigroups of $S$ such that for each $\alpha \in I$, $S_\alpha$ is $A_\alpha$-inner amenable and $A = \bigcup_{\alpha \in I} A_\alpha$ with the following conditions:

(a) for each $S_\alpha, S_\beta$ that are $A_\alpha$-inner amenable and $A_\beta$-inner amenable, respectively, there exists $S_\gamma \supset S_\alpha \cup S_\beta$ such that $S_\gamma$ is $A_\gamma$-inner amenable with $A_\gamma \supset A_\alpha \cup A_\beta$.

(b) $S = \bigcup_{\alpha \in I} S_\alpha$.

Then $S$ is $A$-inner amenable.

Proof. Assume that $h = \sum_{k=1}^n (a_k(f_k) - (f_k)a_k)$ such that $f_k \in \ell^\infty(S)$, $a_k \in A$. By the assumption, there exists a $S_\lambda$ such that $a_k \in A_\lambda$. Since $S_\lambda$ is $A_\lambda$-inner amenable, it follows from Theorem 2.2 that $\sup\{h(x) : x \in S_\lambda\} \geq 0$. In particular, $\sup\{h(x) : x \in S\} \geq 0$. Again by Theorem 2.2, $S$ is $A$-inner amenable. \hfill \Box

Remark 2.5. A subsemigroup of an $A$-inner amenable semigroup need not be $A$-inner amenable. As an example let $S$ be any non $A$-inner amenable semigroup, and let $S^0$ contain $S$ and one new element $o$ such that $os = so = oo = o$, and $S$ is a subsemigroup of $S^0$. Then $S^0$ has an inner $A$-invariant mean: $m(f) = f(o)$, whereas $S$ is not an $A$-inner amenable.

Theorem 2.6. Let $T$ be a subsemigroup of $S$ and $A \subseteq T$. Then $T$ is an $A$-inner amenable if and only if $S$ is an $A$-inner amenable with mean $m$ such that $m(\chi_T) = 1$.

Proof. Let $\theta : T \rightarrow S$ be the embedding map. Then it induces $\vec{\theta} : \ell^\infty(S) \rightarrow \ell^\infty(T)$ by $\vec{\theta}(f) = f|_{T}$. It is easily that $\vec{\theta}$ is bounded and linear. Consider $\vec{\theta}^* : \ell^\infty(T)^* \rightarrow \ell^\infty(S)^*$. Now suppose that $m \in \ell^\infty(S)^*$ is an inner $A$-invariant mean. Clearly $\vec{\theta}^*(m)$ is a mean on $\ell^\infty(S)$. Also, for any $f \in \ell^\infty(S)$, $a \in A$, it is easy to see that

$$\vec{\theta}(af) = (af)|_{T} = a(f|_{T}) = a(\vec{\theta}(f)),$$

and

$$\vec{\theta}(f_a) = (f_a)|_{T} = (f|_{T})_a = (\vec{\theta}(f))_a.$$

Therefore, for all $f \in \ell^\infty(S)$, $a \in A$, we get

$$(\vec{\theta}^*(m))(af) = m(\vec{\theta}(af)) = m(a(\vec{\theta}(f))) = m((\vec{\theta}(f))_a) = (\vec{\theta}^*(m))(f_a).$$

This means that, $\vec{\theta}^*(m)$ is an inner $A$-invariant mean on $\ell^\infty(S)$. Also,

$$\vec{\theta}^*(m)(\chi_T) = m(\vec{\theta}(\chi_T)) = m(\chi_T|_{T}) = m(1) = 1.$$
Conversely, Suppose that $m$ is an inner $A$-invariant mean on $ℓ^∞(S)$ such that $m(χ_T) = 1$. Define the mapping $ϕ : ℓ^∞(T) → ℓ^∞(S)$ by

$$ϕ(f)(t) = \begin{cases} f(t) & t \in T \\ 0 & t \in S \setminus T \end{cases}$$

It is obvious that $ϕ$ is a bounded and linear. Consider $ϕ^* : ℓ^∞(S)^* → ℓ^∞(T)^*$. For any $f \in ℓ^∞(T)$ with $f \geq 0$, we have $ϕ(f) \geq 0$. It is easy to see that $ϕ^*(m)$ is a mean on $ℓ^∞(T)$. Also, for any $f \in ℓ^∞(T)$, $a \in A$ and $t \in T$, we get

$$(ϕ(a) - a(ϕ(f)))(t) = (a(f))(t) - (ϕ(f))(at) = f(at) - f(at) = 0.$$  

So, $(ϕ(a) - a(ϕ(f)))|_T = 0$, and

$$|ϕ(a) - a(ϕ(f))| \leq ||ϕ(a) - a(ϕ(f))||_{A, S \setminus T}.$$  

This implies that $m(ϕ(a) - a(ϕ(f))) = 0$, or, $m(ϕ(a)f) = m(ϕ(f))$. Similarly, one can show that $m(ϕ(f_a)) = m(ϕ(f))$. Therefore,

$$(ϕ^*(m))(a) = m(ϕ(a)f) = m(ϕ(f))$$

$$= m((ϕ(f))a) = m(ϕ(f_a))$$

$$= (ϕ^*(m))(f_a).$$

This shows that, $ϕ^*(m)$ is an inner $A$-invariant mean on $ℓ^∞(T)$.

Given semigroups $S$ and $T$, a map $ϕ : S → T$ is called a homomorphism if it satisfies

$$ϕ(s_1s_2) = ϕ(s_1)ϕ(s_2) (s_1, s_2 \in S).$$

**Theorem 2.7.** Let $S, T$ be semigroups and $ϕ$ be a homomorphism of $S$ onto $T$. If $S$ is $A$-inner amenable, then $T$ is $ϕ(A)$-inner amenable.

**Proof.** Assume that $m$ is an inner $A$-invariant mean on $ℓ^∞(S)$. Put $m_a(f) = m(foϕ)$ for each $f \in ℓ^∞(T)$. Now for every $s \in S, b \in B = ϕ(A)$ and $f \in ℓ^∞(T)$ we have

$$bf_0ϕ(s) = f(bϕ(s)) = f(ϕ(a)ϕ(s)) = f(ϕ(as)) = (foϕ)(as) = a(foϕ)(s),$$

and

$$f_0ϕ(s) = f(ϕ(s)b) = f(ϕ(s)ϕ(a)) = f(ϕ(sa)) = (foϕ)(sa) = (foϕ)a(s).$$

where $a \in A$ is such that $ϕ(a) = b$. So, $bf_0ϕ = a(foϕ)$ and $f_0ϕ = (foϕ)a$. It follows from this relations that

$$m_a(bf) = m(ϕ(foϕ)) = m((foϕ)a) = m(f_0ϕ) = m_0(f).$$

Thus $m_0$ is an inner $ϕ(A)$-invariant mean.

Let $S$ and $T$ be semigroups. Then $S \times T$ is a semigroup with the operation $(s_1,t_1)(s_2,t_2) = (s_1s_2,t_1t_2)$ for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Also we can consider $ℓ^∞(S \times T)$ as a Banach $S \times T$-bimodule via

$$(s,t)f(s',t') = f(ss',tt'), \quad (f(s,t))(s',t') = f(s's,t't),$$

for all $s, s' \in S, t, t' \in T$ and $f \in ℓ^∞(S \times T)$. The homomorphisms $π_S : S \times T → S$ and $π_T : S \times T → T$ with $π_S(s,t) = s, π_T(s,t) = t$, respectively, are called projection homomorphisms.

**Theorem 2.8.** Let $S, T$ be semigroups such that $ℓ^∞(S \times T) = ℓ^∞(S) \times ℓ^∞(T)$. $S$ and $T$ are $A$-inner amenable and $B$-inner amenable, respectively if and only if $S \times T$ is $(A \times B)$-inner amenable.
Proof. Suppose that \( m \) and \( n \) are inner \( A \)-invariant and inner \( B \)-invariant means for \( \ell^\infty(S) \) and \( \ell^\infty(T) \), respectively. Define the mean \( m_o \) on \( \ell^\infty(S \times T) \) by \( m_o(f,g) = m(f) n(g) \) for all \( f \in \ell^\infty(S) \) and \( g \in \ell^\infty(T) \). Then for each \((a,b) \in A \times B\)

\[
m_o((a,b)(f,g)) = m_o(a f, b g) = m(a f) n(b g) \]

\[
= m(f_a) n(g_b) = m_o(f_a, g_b) \\
= m_o((f,g)_{(a,b)}).
\]

This means that \( m_o \) is inner \((A \times B)\)-invariant mean.

Conversely, suppose that \( S \times T \) is \((A \times B)\)-inner amenable. Then by projection homomorphism \( \pi_S(A \times B) = A \) and Theorem 2.7, we obtain that \( S \) is \( A \)-inner amenable. Similarly, we conclude that \( T \) is \( B \)-inner amenable.

**Theorem 2.9.** Let \( S, T \) be two semigroups such that \( S \) and \( T \) are \( A \)-amenable and \( B \)-inner amenable, respectively. Then \( S \times T \) is \((A \times B)\)-inner amenable.

Proof. Suppose that \( m \) be an \( A \)-invariant mean on \( \ell^\infty(S) \) and \( n \) be an inner \( B \)-invariant mean on \( \ell^\infty(T) \). For each \( f \in \ell^\infty(S \times T) \) and \((s,t) \in S \times T\), we consider \( f_T \in \ell^\infty(T) \) and \( f_S^j \in \ell^\infty(S) \) by \( f_S^j(s) = f(s,t) \) and \( f_T(t) = m(f_S^j) \). Now, define the mean \( m_o \) on \( \ell^\infty(S \times T) \) by

\[
m_o(f) = n(f_T) \text{ for all } f \in \ell^\infty(S \times T).
\]

For every \((a,b) \in A \times B \) it follows that \( (f_S^j)_{(a,b)} = a(f_S^j) \) and \( (f_S^j)_{(a,b)} = (f_S^j)_{(a,b)} b \). Furthermore, for every \( t \in T \)

\[
((a,b)f)_T(t) = m((a,b)f_T) = m(a(f_T)) \\
= m(f_S^j) = (f_T(bt)) \\
= b(f_T)(t).
\]

That is, \((a,b)f)_T = b(f_T)\). Similarly, one find that \((f_T)(a,b) = (f_T)(a,b)\).

For every \( f \in \ell^\infty(S \times T) \) and \((a,b) \in A \times B \) we get

\[
m_o((a,b)f) = n((a,b)f_T) = n(b(f_T)) \\
= n(f_T) = n((f_T)b) \\
= m_o(f_{(a,b)}).
\]

It follows that \( m \) is an inner \((A \times B)\)-invariant mean. \( \square \)

3. Examples of \( A \)-inner amenability

**Example 3.1.** If there exists an element \( x \) in semigroup \( S \) that commutes with all \( a \in A \), then the Dirac measure \( \delta_x \) for all \( f \in \ell^\infty(S) \) is an inner \( A \)-invariant mean on \( \ell^\infty(S) \).

\[
\delta_x(a f) = f(ax) = f(xa) = \delta_x(f_a).
\]

In the following examples, we study \( A \)-inner amenability over a left (right) zero semigroup, that is a semigroup whose multiplication is defined by \( st = s \) \((st = t)\) for all \( s, t \in S \). We denote the cardinal number of a set \( A \) by \(|A|\).

**Example 3.2.** Let \( S \) be a left zero semigroup, then for any subset \( A \) of \( S \):

(i) if \(|A| = 1\), then \( S \) is \( A \)-inner amenable;

(ii) if \(|A| \geq 2\), then \( S \) is not \( A \)-inner amenable.
Proof. (i) Assume that \( A = \{a\} \). Define \( m \in \ell^\infty(S)^* \) by \( m(f) = f(a) \) for every \( f \in \ell^\infty(S) \). Then we obtain \( m(a_f) = a f(a) = f(aa) = f_a(a) = m(f_a) \). This shows that \( S \) is \( A \)-inner amenable.

(ii) Clearly for each \( a \in A \) we have
\[
a f = f(a) \quad \text{and} \quad f_a = f.
\]

Now, if we suppose that \( S \) is \( A \)-inner amenable with an inner \( A \)-invariant mean \( m \), then for every \( a \in A \) and \( f \in \ell^\infty(S) \), we have \( m(a_f) = m(f_a) \). Therefore \( f(a) = m(f) \). Now if we consider \( a \neq b \in A \) and \( f = \chi_{\{a\}} \) then we obtain
\[
1 = f(a) = m(f) = f(b) = 0.
\]

This is a contradiction. \( \square \)

Example 3.3. Let \( \mathbb{F}_2 \) be free group on two generators \( a \) and \( b \). If \( A \) is the set of elements of \( \mathbb{F}_2 \) that begin with \( a \) or \( a^{-1} \) when written as reduced words. Then \( \mathbb{F}_2 \) is not \( A \)-inner amenable.

Proof. We consider \( f = \chi_A \) and
\[
h = ((ba^{-1}f)ab^{-1} - ab^{-1}(ba^{-1}f)) + ((b^{-1}a^{-1}f)aba - aba(b^{-1}a^{-1}f)).
\]

Clearly \( h \in \mathcal{H}(A) \). Now by Theorem 2.2 it is enough to prove that the function \( h \) has the property that, \( \sup \{h(x) : x \in \mathbb{F}_2\} < 0 \). For each \( x \in \mathbb{F}_2 \) we have
\[
h(x) = f(ba^{-1}xab^{-1}) + f(b^{-1}a^{-1}xaba) - f(ax) - f(x).
\]

Now the argument as in the proof of Theorem (17.16) of [7] shows \( \sup \{h(x) : x \in \mathbb{F}_2\} \leq -1 \). \( \square \)

By use of Theorem 2.2 in the following example, we study \( A \)-inner amenability over a right zero semigroup.

Example 3.4. Let \( S \) be a right zero semigroup, then for any subset \( A \) of \( S \) we have

(i) If \( |A| = 1 \), then \( S \) is \( A \)-inner amenable.

(ii) If \( |A| \geq 2 \), then \( S \) is not \( A \)-inner amenable.

Proof. (i) Assume that \( A = \{a\} \). Since for every \( h \in \mathcal{H}(A) \) and \( x \in S \) we have
\[
h(x) = \sum_{k=1}^{n} ((f_k)_a - a(f_k))(x)
\]
\[
= \sum_{k=1}^{n} (f_k(xa) - f_k(ax))
\]
\[
= \sum_{k=1}^{n} (f_k(a) - f_k(x)).
\]

Then by set \( x = a \) we have \( \sup \{h(x) : x \in S\} \geq 0 \). This shows that \( S \) is \( A \)-inner amenable.

(ii) For \( a \neq b \in A \), we take \( h = (a(\chi_{\{a\}}) - (\chi_{\{a\}})a) + (b(\chi_{\{b\}}) - (\chi_{\{b\}})b) \). Hence for each \( x \in S \) we obtain
\[
h(x) = (a(\chi_{\{a\}}) - (\chi_{\{a\}})a)(x) + (b(\chi_{\{b\}}) - (\chi_{\{b\}})b)(x)
\]
\[
= (\chi_{\{a\}}(ax) - (\chi_{\{a\}})(xa)) + (\chi_{\{b\}}(bx) - (\chi_{\{b\}})(xb))
\]
\[
= (\chi_{\{a\}}(x) + \chi_{\{b\}}(x) - 2.
\]

and this implise that \( \sup \{h(x) : x \in S\} \leq -1 \). Hence by theorem 2.2 \( S \) is not \( A \)-inner amenable. \( \square \)
4. Følner’s condition

Before stating the following theorem, recall that a mean in $\ell^1(S)$ is called a finite mean if it is a convex combination of the Dirac measures. We shall use $\Phi$ denote the set of all finite means and $\delta_x$ denotes the Dirac measure at $x \in S$. It is obvious that $\Phi$ is convex subset of $\ell^1(S)$. In fact, $\Phi$ is convex hull of $S$.

**Theorem 4.1.** Let $S$ be a semigroup and $A \subseteq S$. Then the following statements are equivalent:

(a) $S$ is $A$-inner amenable.

(b) there is a net $(\varphi_\alpha)$ of finite means such that $\delta_a \ast \varphi_\alpha - \varphi_\alpha \ast \delta_a \rightarrow 0$ in the weak topology of $\ell^1(S)$, for every $a \in A$.

(c) there is a net $(\psi_\alpha)$ of finite means such that $\|\delta_a \ast \psi_\alpha - \psi_\alpha \ast \delta_a\|_1 \rightarrow 0$ for every $a \in A$.

Proof. (a) $\Rightarrow$ (b). Let $m$ be an inner $A$-invariant mean on $\ell^\infty(S)$. Since $m \in \ell^\infty(S)^*$, we can find a net $(\varphi_\alpha)$ of finite means such that $\lim_{\alpha} \varphi_\alpha = m$ in the weak* topology of $\ell^\infty(S)^*$. Then for all $f \in \ell^\infty(S)$ and $a \in A$,

$$f(\delta_a \ast \varphi_\alpha - \varphi_\alpha \ast \delta_a) = \varphi_\alpha(af) - \varphi_\alpha(fa) \rightarrow m(af) - m(fa) = 0.$$ 

It follows that $\delta_a \ast \varphi_\alpha - \varphi_\alpha \ast \delta_a \rightarrow 0$ in the weak topology of $\ell^1(S)$, for every $a \in A$.

(b) $\Rightarrow$ (c). Let $(\varphi_\beta)$ be a net as in (b). Using an idea of Ling [11], we define linear map $T : \ell^1(S) \rightarrow \prod_{a \in A} \ell^1(S)$ by $T(\varphi)(a) = \delta_a \ast \varphi - \varphi \ast \delta_a$ for every $\varphi \in \ell^1(S), a \in A$. Now by assumption, $T(\varphi_\beta)(a) = \delta_a \ast \varphi_\beta - \varphi_\beta \ast \delta_a \rightarrow 0$ weakly in $\ell^1(S)$, for every $a \in A$. This means that zero lies in the weak closure of $T(\Phi)$. Since $\prod_{a \in A} \ell^1(S)$ with product of the norm topology is a locally convex space and $\Phi$ is convex, the closure of $T(\Phi)$ in this topology contains 0. Thus, there exists a subnet $(\varphi_\alpha) \subseteq (\varphi_\beta)$ such that $\lim_{\alpha} \delta_a \ast \psi_\alpha - \psi_\alpha \ast \delta_a\|_1 \rightarrow 0$ for every $a \in A$.

(c) $\Rightarrow$ (a). Since convergence in norm implies convergence in weak topology, this implication is trivial.

(b) $\Rightarrow$ (a). Let $(\varphi_\alpha)$ be a net satisfying the convergence in (b). By Alaoglu’s theorem, it has a weak* convergent subnet. By passing to such a subnet if necessary, there is a $m \in \ell^\infty(S)^*$ such that $\lim_{\alpha} \varphi_\alpha = m$ in the weak* topology of $\ell^\infty(S)^*$. Therefore $m$ is a mean on $\ell^\infty(S)$, and for all $a \in A, f \in \ell^\infty(S)$

$$m(af) - m(fa) = \lim_{\alpha} (\varphi_\alpha(af) - \varphi_\alpha(fa)) = \lim_{\alpha} (\delta_a \ast \varphi_\alpha - \varphi_\alpha \ast \delta_a)(f) = 0.$$ 

\]

For each $s \in S$ we put $s^{-1}A = \{t \in S : st \in A\}$ and $As^{-1} = \{t \in S : ts \in A\}$. We also note that $\frac{1}{|A|}\chi_A$ defines an element in $\ell^1(S)$.

**Lemma 4.2.** Let $A$ acts injectively on the right of semigroup $S$, then for every $B \subseteq A$ and $a \in A$

$$||\chi_B \ast \delta_a - \delta_a \ast \chi_B||_1 = 2|Ba \setminus aB|.$$ 

Proof. For $a \in A$ and $B \subseteq A$, we get

$$(\delta_a \ast \chi_B)(x) = \sum_{as=x} \chi_B(s)$$

$$= \sum_{s \in a^{-1}\{x\}} \chi_B(s)$$

$$= |B \cap a^{-1}\{x\}|.$$ 

Similarly, we obtain $(\chi_B \ast \delta_a)(x) = |B \cap \{x\}a^{-1}|$. It is easy to see that

$$(\chi_B \ast \delta_a - \delta_a \ast \chi_B)(x) = \begin{cases} |B \cap \{x\}a^{-1}| & \text{if } x \in Ba \setminus aB \\ -|B \cap a^{-1}\{x\}| & \text{if } x \in aB \setminus Ba \\ |B \cap \{x\}a^{-1}| - |B \cap a^{-1}\{x\}| & \text{if } x \in aB \cap Ba \\ 0 & \text{if } x \notin aB \cup Ba \end{cases}$$
Since $A$ acts injectively on the right of semigroup $S$, then for each $x \in Ba$ we obtain $|B \cap \{x\}a^{-1}| = 1$. This implies that

$$
\|\chi_{B} \ast \delta_{a} - \delta_{a} * \chi_{B}\|_{1} = \sum_{x \in Ba \setminus aB} 1 + \sum_{x \in aB \setminus Ba} |B \cap a^{-1}\{x\}| + \sum_{x \in aB \cap Ba} (|B \cap a^{-1}\{x\}| - 1)
$$

$$
= |Ba \setminus aB| + \sum_{x \in Ba} |B \cap a^{-1}\{x\}| - |aB \cap Ba|
$$

$$
= |Ba \setminus aB| + |B| - |aB \cap Ba|
$$

$$
= |Ba \setminus aB| + |Ba \setminus aB|
$$

$$
= 2|Ba \setminus aB|
$$

\[\square\]

**Theorem 4.3.** Let $A$ act injectively on the right of semigroup $S$. If for any finite set $F \subseteq A$ and any $\varepsilon > 0$, there exists a finite non-empty set $B \subseteq A$ such that $|Ba \setminus aB| < \varepsilon|B|$ for all $a \in F$, then $S$ is $A$-inner amenable.

**Proof.** By the assumption there exists a net of finite non-empty sets $B_{\alpha} \subseteq A$ such that

$$
|B_{\alpha} \setminus aB_{\alpha}|/|B_{\alpha}| \longrightarrow 0 \quad \text{for all} \quad a \in A.
$$

By Lemma 4.2 we have

$$
\|\chi_{B_{\alpha}} \ast \delta_{a} - \delta_{a} * \chi_{B_{\alpha}}\|_{1} = |B_{\alpha} \setminus aB_{\alpha}|.
$$

Set $\varphi_{a} = |B_{\alpha}|^{-1} \chi_{B_{\alpha}}$. Then for $\alpha$, and $a \in A$

$$
\|\delta_{a} * \varphi_{a} - \varphi_{a} * \delta_{a}\|_{1} \longrightarrow 0.
$$

Now the proof is complete by Theorem 4.1. \[\square\]

**Remark 4.4.** The assumption of Theorem 4.3 ‘that $A$ acts injectively on the right of semigroup $S$’ is necessary. In fact, any right zero semigroup $S$ is not $A$-inner amenable if $A$ has at least two elements (see Example 3.2).

**References**