Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Non-Instantaneous Impulses in Banach Spaces

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Abstract
In the present article, we prove some results concerning the existence of solutions for a class of initial value problem for nonlinear implicit fractional differential equations with non-instantaneous impulses and generalized Hilfer fractional derivative in Banach spaces. The results are based on fixed point theorems of Darbo and Mönch associated with the technique of measure of noncompactness. An example is included to show the applicability of our results.

Keywords: Implicit fractional differential equations, generalized Hilfer fractional derivative, initial value problem, existence, measure of noncompactness, fixed point, non-instantaneous impulses, Ulam stability, Banach space.

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1. Introduction
Fractional derivatives and fractional integrals generalize to noninteger order the derivative and the integral of a function. There are several kinds of fractional derivatives, such as, the Riemann–Liouville fractional derivative, the Grunwald–Letnikov fractional derivative, the Caputo derivative, the Marchaud fractional derivative, the generalized Hilfer derivative, etc. \cite{4,5,6,16,17}. There are numerous books and articles

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focused on linear and nonlinear problems for fractional differential equations involving different kinds of fractional derivatives, see, for example, [3] [7] [8] [10] [11] [13].

The class of problems for fractional differential equations with abrupt and instantaneous impulses is vastly studied, and different topics on the existence and qualitative properties of solutions are considered, [13] [15] [28]. In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of drugs into the bloodstream and the consequent absorption for the body are a gradual and continuous process. In [11] [2] [9] the authors studied some new classes of abstract impulsive differential equations with not instantaneous impulses.

The stability of functional equations originated from a question by Ulam [27]. Hyers [21] gave a first affirmative partial answer to Ulam’s question for Banach spaces. Hyers Theorem was generalized by Rassias [23] in 1978. Afterwards, many interesting results of the generalized Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations have been investigated by a number of mathematicians; one can see the monograph of Abbas et al. [8] and the paper by Rus [24] in which the Ulam-Hyers stability for operator equations is discussed.

Motivated by the works mentioned above, in this paper, we establish existence results for the initial value problem of a nonlinear implicit generalized Hilfer-type fractional differential equation with non-instantaneous impulses,

\[
\left( \rho D_{s_k}^{\alpha,\beta} u \right)(t) = f \left( t, u(t), \left( \rho D_{s_k}^{\alpha,\beta} u \right)(t) \right); \ t \in I_k, \ k = 0, \ldots, m, \quad (1)
\]

\[ u(t) = g_k(t, u(t)); \ t \in \bar{I}_k, \ k = 1, \ldots, m, \quad (2) \]

\[ \left( \rho J_{a^+}^{1-\gamma} u \right)(a^+) = \phi_0, \quad (3) \]

where \( \rho D_{s_k}^{\alpha,\beta} \) and \( \rho J_{a^+}^{1-\gamma} \) are, respectively, the generalized Hilfer-type fractional derivative of order \( \alpha \in (0, 1) \) and type \( \beta \in (0, 1) \) and generalized fractional integral of order \( 1 - \gamma \) (\( \gamma = \alpha + \beta - \alpha \beta \)), \( \rho > 0 \), \( \phi_0 \in E \), \( I_k := (s_k, t_{k+1}] \); \( k = 0, \ldots, m \), \( \bar{I}_k := (t_k, s_k] \); \( k = 1, \ldots, m \), \( a = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \ldots < s_{m-1} < t_m \leq s_m < t_{m+1} = b < \infty \), \( u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon) \) and \( u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon) \) represent the right and left hand limits of \( u(t) \) at \( t = t_k \), \( f : I_k \times E \times E \to E \) is a given function and \( g_k : \bar{I}_k \times E \to E; k = 1, \ldots, m \), are given continuous functions such that \( \left( \rho J_{s_k}^{1-\gamma} g_k \right)(t, u(t)) |_{t=s_k} = \phi_k \in E \), where \((E, \| \cdot \|)\) is a real Banach space.

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminaries about the generalized Hilfer fractional derivative and auxiliary results. In Section 3, two results for the problem (1)-(3) are presented which are based on the fixed point theorems of Mönch and Darbo associated with the technique of measure of noncompactness. In Section 4, we discuss the Ulam-Hyers-Rassias Stability for the problems. Finally, in the last section, we give an example to illustrate the applicability of our results.

### 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let \( 0 < a < b, J = [a, b] \). Let \((E, \| \cdot \|)\) be a Banach space.

By \( C \) we denote the Banach space of all continuous functions from \( J \) into \( E \) with the norm

\[ \| u \|_\infty = \sup \{ \| u(t) \| : t \in J \}. \]
Consider the weighted Banach space
\[
C_{\gamma,\rho}(I_k) = \left\{ u : I_k \to E : t \to \left( \frac{t^\rho - s_k^\rho}{\rho} \right)^{1-\gamma} u(t) \in C([s_k, t_{k+1}], E) \right\},
\]
where \( 0 \leq \gamma < 1, k = 0, \ldots, m, \) and
\[
C^n_{\gamma,\rho}(I_k) = \left\{ u \in C^{n-1}(I_k) : u^{(n)} \in C_{\gamma,\rho}(I_k) \right\}, n \in \mathbb{N},
\]
\[
C^0_{\gamma,\rho}(I_k) = C_{\gamma,\rho}(I_k).
\]
Also consider the Banach space
\[
PC_{\gamma,\rho}(J) = \left\{ u : (a, b] \to E : u \in C_{\gamma,\rho}(\bigcup_{k=0}^m I_k) \cap C(\bigcup_{k=1}^m I_k, E) \right\},
\]
where \( u(t_k^-), u(t_k^+), u(s_k^-), \) and \( u(s_k^+) \) with \( u(t_k^-) = u(t_k) \), \( 0 \leq \gamma < 1, \)
\[
PC^n_{\gamma,\rho}(J) = \left\{ u \in PC^{n-1}(J) : u^{(n)} \in PC_{\gamma,\rho}(J) \right\}, n \in \mathbb{N},
\]
\[
PC^0_{\gamma,\rho}(J) = PC_{\gamma,\rho}(J),
\]
with the norm
\[
\|u\|_{PC_{\gamma,\rho}} = \max \left\{ \max_{k=0, \ldots, m} \left\{ \sup_{t \in I_k} \left\| \left( \frac{t^\rho - s_k^\rho}{\rho} \right)^{1-\gamma} u(t) \right\| \right\}, \max_{k=1, \ldots, m} \left\{ \sup_{t \in I_k} \|u(t)\| \right\} \right\}.
\]
By \( L^1(J) \), we denote the space of Bochner-integrable functions \( f : J \to E \) with the norm
\[
\|f\|_1 = \int_a^b \|f(t)\| dt.
\]

**Definition 2.1.** [22] (Generalized fractional integral).
Let \( \alpha \in \mathbb{R}_+ \) and \( g \in L^1(J) \). The generalized fractional integral of order \( \alpha \) is defined by
\[
\left( ^\rho J^\alpha_{a^+} g \right)(t) = \int_a^t s^{\rho-1} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \ t > a, \rho > 0,
\]
where \( \Gamma(\cdot) \) is the Euler gamma function defined by \( \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \ \alpha > 0. \)

**Definition 2.2.** [22] (Generalized fractional derivative).
Let \( \alpha \in \mathbb{R}_+ \setminus \mathbb{N} \) and \( \rho > 0 \). The generalized fractional derivative \( ^\rho D^\alpha_{a^+} \) of order \( \alpha \) is defined by
\[
\left( ^\rho D^\alpha_{a^+} g \right)(t) = \delta^\alpha_{\rho} \left( ^\rho J^{\alpha-\alpha}_{a^+} g \right)(t)
\]
\[
= \left( t^{1-\rho} \frac{d}{dt} \right)^n \int_a^t s^{\rho-1} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds, t > a,
\]
where \( n = [\alpha] + 1 \) and \( \delta^\alpha_{\rho} = \left( t^{1-\rho} \frac{d}{dt} \right)^n \).

**Theorem 2.1.** [22] Let \( \alpha > 0, \beta > 0, 1 \leq p \leq \infty, 0 < a < b < \infty. \) Then, for \( g \in L^1([s_k, t_{k+1}]), k = 0, \ldots, m, \) we have
\[
\left( ^\rho J^\alpha_{s_k} \left( ^\rho J^\beta_{s_k} g \right) \right)(t) = \left( ^\rho J^\alpha_{s_k} g \right)(t).
\]
Lemma 2.1. [22, 24] Let $\alpha > 0$, $0 \leq \gamma < 1$ and $k = 0, \ldots, m$. Then, $D^{\alpha}_{s_k^+} \mathcal{J}^\gamma_{s_k}$ is bounded from $C_{\gamma, \rho}(I_k)$ into $C_{\gamma, \rho}(I_k)$.

Lemma 2.2. [24] Let $0 < a < b < \infty$, $\alpha > 0$, $0 \leq \gamma < 1$, $u \in C_{\gamma, \rho}(I_k)$ and $k = 0, \ldots, m$. If $\alpha > 1 - \gamma$, then $\rho \mathcal{J}^\alpha_{s_k^+} u \in C([s_k, t_{k+1}], E)$ and

$$(\rho \mathcal{J}^\alpha_{s_k^+} u)(s_k) = \lim_{t \to s_k^+} (\rho \mathcal{J}^\alpha_{s_k^+} u)(t) = 0.$$ 

Lemma 2.3. [24] Let $t > s_k$, $k = 0, \ldots, m$. Then, for $\alpha \geq 0$ and $\beta > 0$, we have

$$
\left[ \rho \mathcal{J}_s^\alpha \left( \frac{s^\rho - s_k^\rho}{\rho} \right) \right]^{\beta - 1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha + \beta)} \left( \frac{t^\rho - s_k^\rho}{\rho} \right)^{\alpha + \beta - 1},
$$

$$
\left[ \rho \mathcal{D}_s^\alpha \left( \frac{s^\rho - s_k^\rho}{\rho} \right) \right]^{\alpha - 1}(t) = 0, \quad 0 < \alpha < 1.
$$

Lemma 2.4. [24] Let $\alpha > 0$, $0 \leq \gamma < 1$, $k = 0, \ldots, m$, and $g \in C_{\gamma, \rho}(I_k)$. Then,

$$
\left( \rho \mathcal{D}^\alpha_{s_k^+} \rho \mathcal{J}^\alpha_{s_k^+} g \right)(t) = g(t), \quad \text{for all } t \in I_k, k = 0, \ldots, m.
$$

Lemma 2.5. [24] Let $0 < \alpha < 1$, $0 \leq \gamma < 1$, $k = 0, \ldots, m$. If $g \in C_{\gamma, \rho}(I_k)$ and $\rho \mathcal{J}^{1-\alpha}_{s_k^+} g \in C^1_{\gamma, \rho}(I_k)$, then for all $t \in I_k, k = 0, \ldots, m$,

$$
\left( \rho \mathcal{J}^\alpha_{s_k^+} \rho \mathcal{D}^\alpha_{s_k^+} g \right)(t) = g(t) - \left( \rho \mathcal{J}^{1-\alpha}_{s_k^+} g \right)(s_k) \left( \frac{t^\rho - s_k^\rho}{\rho} \right)^{\alpha - 1}.
$$

Definition 2.3. [24] Let order $\alpha$ and type $\beta$ satisfy $n - 1 < \alpha < n$ and $0 \leq \beta \leq 1$, with $n \in \mathbb{N}$, and $k = 0, \ldots, m$. The generalized Hilfer-type fractional derivative, with $\rho > 0$ of a function $g \in C_{\gamma, \rho}(I_k)$, is defined by

$$
\left( \rho \mathcal{D}^{\alpha, \beta}_{s_k^+} g \right)(t) = \left( \rho \mathcal{J}^{(n-\alpha)}_{s_k^+} \left( \frac{t^\rho - s_k^\rho}{\rho} \right)^n \rho \mathcal{J}^{(1-\beta)(n-\alpha)}_{s_k^+} g \right)(t) = \left( \rho \mathcal{J}^{(\alpha-\beta)(n-\alpha)}_{s_k^+} g \right)(t).
$$

In this paper we consider the case $n = 1$ only, because $0 < \alpha < 1$.

Property 2.2. [24] The operator $\rho \mathcal{D}^{\alpha, \beta}_{s_k^+}$ can be written as

$$
\rho \mathcal{D}^{\alpha, \beta}_{s_k^+} = \rho \mathcal{J}^{\beta(1-\alpha)}_{s_k^+} \delta_\rho \rho \mathcal{J}^{\gamma-\alpha}_{s_k^+} = \rho \mathcal{J}^{\beta(1-\alpha)}_{s_k^+} \rho \mathcal{D}^{\gamma}_{s_k^+}, \quad \gamma = \alpha + \beta - \alpha \beta, \ k = 0, \ldots, m.
$$

Property 2.3. [24] The fractional derivative $\rho \mathcal{D}^{\alpha, \beta}_{s_k^+}$ is an interpolator of the following fractional derivatives:

Hilfer ($\rho \to 1$), Hilfer–Hadamard ($\rho \to 0^+$), generalized ($\beta = 0$), Caputo–type ($\beta = 1$), Riemann–Liouville ($\beta = 0, \rho \to 1$), Hadamard ($\beta = 0, \rho \to 0^+$), Caputo ($\beta = 1, \rho \to 1$), Caputo–Hadamard ($\beta = 1, \rho \to 0^+$), Liouville ($\beta = 0, \rho \to 1, \alpha = 0$) and Weyl ($\beta = 0, \rho \to 1, \alpha = -\infty$).

Consider the following parameters $\alpha, \beta, \gamma$ satisfying

$$\gamma = \alpha + \beta - \alpha \beta, \quad 0 < \alpha, \beta, \gamma < 1.$$
We define the spaces
\[
C_{\gamma, \varrho}^{\alpha, \beta}(I_k) = \left\{ u \in C_{\gamma, \varrho}(I_k), \, \varrho D_{s_k}^{\alpha, \beta} u \in C_{\gamma, \varrho}(I_k) \right\},
\]
and
\[
C_{\gamma, \varrho}^\gamma(I_k) = \left\{ u \in C_{\gamma, \varrho}(I_k), \, \varrho D_{s_k}^\gamma u \in C_{\gamma, \varrho}(I_k) \right\},
\]
where \( k = 0, \ldots, m \).

Since \( \varrho D_{s_k}^{\alpha, \beta} u = \varrho J_{s_k}^{\gamma(1-\alpha)} \varrho D_{s_k}^\gamma u \), it follows from Lemma 2.1 that
\[
C_{\gamma, \varrho}^\gamma(I_k) \subset C_{\gamma, \varrho}^{\alpha, \beta}(I_k) \subset C_{\gamma, \varrho}(I_k).
\]

Also,
\[
PC_{\gamma, \varrho}^\gamma(J) = \left\{ u : (a, b) \to E : u \in C_{\gamma, \varrho}(\bigcup_{k=0}^m I_k) \cap C(\bigcup_{k=1}^m I_k, E) \right\}.
\]

Lemma 2.6. \([24]\) Let \( 0 < \alpha < 1, 0 \leq \beta \leq 1, \, \gamma = \alpha + \beta - \alpha \beta \) and \( k = 0, \ldots, m \). If \( u \in C_{\gamma, \varrho}^\gamma(I_k) \), then
\[
\varrho J_{s_k}^\gamma \varrho D_{s_k}^\gamma u = \varrho J_{s_k}^\alpha \varrho D_{s_k}^{\alpha, \beta} u,
\]
and
\[
\varrho D_{s_k}^\gamma \varrho J_{s_k}^\gamma u = \varrho D_{s_k}^{\beta(1-\alpha)} u.
\]

Definition 2.4. \([24]\) Let \( X \) be a Banach space and let \( \Omega_X \) be the family of bounded subsets of \( X \). The Kuratowski measure of noncompactness is the map \( \mu : \Omega_X \to [0, \infty) \) defined by
\[
\mu(M) = \inf\{ \epsilon > 0 : M \subset \bigcup_{j=1}^m M_j, \text{diam}(M_j) \leq \epsilon \},
\]
where \( M \in \Omega_X \). The map \( \mu \) satisfies the following properties:

\begin{itemize}
  \item \( \mu(M) = 0 \iff M \) is compact (\( M \) is relatively compact).
  \item \( \mu(M) = \mu(M) \).
  \item \( M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2) \).
  \item \( \mu(M_1 + M_2) \leq \mu(B_1) + \mu(B_2) \).
  \item \( \mu(cM) = |c| \mu(M), \, c \in \mathbb{R} \).
  \item \( \mu(\text{conv}M) = \mu(M) \).
\end{itemize}

Lemma 2.7. \([19]\) Let \( D \subset PC_{\gamma, \varrho}(J) \) be a bounded and equicontinuous set, then

(i) the function \( t \to \mu(D(t)) \) is continuous on \( J \), and

\[
\mu_{PC_{\gamma, \varrho}} = \max \left\{ \max_{k=0}^m \left\{ \sup_{t \in I_k} \left( \frac{1}{t^\rho - s_k^\rho} \right)^{1-\gamma} u(t) \right\} \right\},
\]

(ii) \( \mu \left( \int_a^b u(s) ds : u \in D \right) \leq \int_a^b \mu(D(s)) ds \), where

\[
D(t) = \{ u(t) : t \in D \}, \, t \in J.
\]
Lemma 2.8. Let $f : I_k \times E \to E$ be a function such that $f(\cdot, u(\cdot), t^{\alpha,\beta} u(\cdot)) \in C(I_k, E), k = 0, \ldots, m$, for any $u \in C_{\gamma, \rho}(I_k)$. Then $u \in C_{\gamma, \rho}(I_k)$ is a solution of the differential equation, for $0 < \alpha < 1, 0 \leq \beta \leq 1$,

$$\left( t^{\alpha,\beta} u \right) (t) = f(t, u(t), t^{\alpha,\beta} u(t)), \text{ for each } t \in I_k, k = 0, \ldots, m, \tag{4}$$

if and only if $u$ satisfies the following Volterra integral equation,

$$u(t) = \left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right) ds, \tag{5}$$

where $\gamma = \alpha + \beta - \alpha \beta$.

**Proof.** Assume $u \in C_{\gamma, \rho}(I_k)$ satisfies the equation (4) where $k = 0, \ldots, m$. We prove that $u$ is a solution to the equation (5). From the definition of the space $C_{\gamma, \rho}(I_k)$ and by using Lemma 2.1 and Definition 2.2, we have

$$\left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right) ds, \tag{5}$$

where $\gamma = \alpha + \beta - \alpha \beta$.

By the definition of the space $C_{\gamma, \rho}(I_k)$, we obtain

$$\left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right) ds, \tag{5}$$

where $\gamma = \alpha + \beta - \alpha \beta$.

Hence, Lemma 2.5 implies that for all $t \in I_k, k = 0, \ldots, m$,

$$\left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right) ds, \tag{5}$$

Using Lemma 2.6 we have

$$\left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right) ds, \tag{5}$$

Therefore,

$$u(t) = \left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right) ds, \tag{5}$$

where $t \in I_k, k = 0, \ldots, m$, that is, $u$ satisfies the equation (5).

Conversely, let $u \in C_{\gamma, \rho}(I_k)$ satisfy the equation (5) where $k = 0, \ldots, m$. We prove that $u$ is a solution to the equation (4). Apply operator $t^{\alpha,\beta}$ on both sides of (5), where $k = 0, \ldots, m$. Then, from Lemma 2.3 and Lemma 2.6, we obtain

$$\left( t^{\alpha,\beta} u \right)(t) = \left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right)(t), \tag{6}$$

Since $u \in C_{\gamma, \rho}(I_k)$ and by definition of $C_{\gamma, \rho}(I_k)$, we have $t^{\alpha,\beta} u \in C_{\gamma, \rho}(I_k)$, then (6) implies that

$$\left( t^{\alpha,\beta} u \right)(t) = \left( \frac{t^{\alpha,\beta}}{\Gamma(\gamma)} \right) \left( \frac{t^\rho - s^\rho}{\rho} \right)^{1-\gamma} \left( s^{\rho-1} f(s, u(s), t^{\alpha,\beta} u(s)) \right)(t), \tag{7}$$

where $t \in I_k, k = 0, \ldots, m$, that is, $u$ satisfies the equation (5).
As \( f(\cdot, u(\cdot), \rho D_{s_k}^{\alpha,\beta} u(\cdot)) \in C(I_k, E) \) and from Lemma 2.1, it follows that

\[
\left( \rho J_{s_k}^{1-\beta(1-\alpha)} f \right) \in C_{\gamma,\rho}(I_k), k = 0, \ldots, m.
\]  

From (7), (8) and by the definition of the space \( C_{\gamma,\rho}(I_k) \), we obtain

\[
\left( \rho J_{s_k}^{1-\beta(1-\alpha)} f \right) \in C_{\gamma,\rho}(I_k), k = 0, \ldots, m.
\]

Applying operator \( \rho J_{s_k}^{\beta(1-\alpha)} \) on both sides of (7) and using Lemma 2.5, Lemma 2.2 and Property 2.2, we have

\[
\begin{align*}
(\rho D_{s_k}^{\alpha,\beta} u)(t) &= \rho J_{s_k}^{\beta(1-\alpha)} \left( \rho D_{s_k}^{\gamma} u \right)(t) \\
&= f(t, u(t), \rho D_{s_k}^{\alpha,\beta} u(t)) \\
&- \frac{\left( \rho J_{s_k}^{1-\beta(1-\alpha)} f \right)(s_k)}{\Gamma(\beta(1-\alpha))} \left( \frac{t^\rho - s_k^\rho}{\rho} \right)^{\beta(1-\alpha)-1} \\
&= f(t, u(t), \rho D_{s_k}^{\alpha,\beta} u(t)),
\end{align*}
\]

that is, (4) holds. This completes the proof. \( \square \)

**Theorem 2.4.** (Mönch’s fixed point Theorem ([20])). Let \( D = \text{a closed, bounded and convex subset of a Banach space } X \text{ such that } 0 \in D, \text{ and let } T \text{ be a continuous mapping of } D \text{ into itself. If the implication}

\[
V = \overline{\text{cont}T}(V), \text{ or } V = T(V) \cup \{0\} \Rightarrow \mu(V) = 0,
\]

holds for every subset \( V \) of \( D \), then \( T \) has a fixed point.

**Theorem 2.5.** (Darbo’s fixed point Theorem ([20])). Let \( D \) be a non-empty, closed, bounded and convex subset of a Banach space \( X \), and let \( T \) be a continuous mapping of \( D \) into itself such that for any non-empty subset \( C \) of \( D \),

\[
\mu(T(C)) \leq k\mu(C),
\]

where \( 0 \leq k < 1 \), and \( \mu \) is the Kuratowski measure of noncompactness on \( X \). Then \( T \) has a fixed point in \( D \).

Now, we consider the Ulam stability for problem (1)–(3) that will be used in Section 4. Let \( u \in PC_{\gamma,\rho}(J) \), \( \epsilon > 0 \), \( \tau > 0 \) and \( \vartheta : (a, b) \to [0, \infty) \) be a continuous function. We consider the following inequality:

\[
\left\| (\rho D_{s_k}^{\alpha,\beta} u)(t) - f \left( t, u(t), \rho D_{s_k}^{\alpha,\beta} u \right)(t) \right\| \leq \epsilon \vartheta(t), t \in I_k, k = 0, \ldots, m,
\]

\[
\left( \left\| u(t) - g_k(t, u(t)) \right\| \leq \epsilon \tau, t \in I_k, k = 1, \ldots, m. \right)
\]

**Definition 2.5.** Problem (1)–(3) is Ulam-Hyers-Rassias (U-H-R) stable with respect to \( (\vartheta, \tau) \) if there exists a real number \( \alpha_f > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( u \in PC_{\gamma,\rho}(J) \) of inequality (11) there exists a solution \( w \in PC_{\gamma,\rho}(J) \) of (1)–(3) with

\[
\left\| u(t) - w(t) \right\| \leq \epsilon a_{f,\rho} \vartheta(t) + \tau, \quad t \in J.
\]

**Remark 2.1.** A function \( u \in PC_{\gamma,\rho}(J) \) is a solution of inequality (11) if and only if there exist \( \sigma \in PC_{\gamma,\rho}(J) \) and a sequence \( \sigma_k, k = 0, \ldots, m, \) such that

1. \( \| \sigma(t) \| \leq \epsilon \vartheta(t), t \in I_k, k = 0, \ldots, m, \) and \( \| \sigma_k \| \leq \epsilon \tau, t \in I_k, k = 1, \ldots, m, \)
2. \( \left( \rho D_{s_k}^{\alpha,\beta} u \right)(t) = f \left( t, u(t), \rho D_{s_k}^{\alpha,\beta} u \right)(t) + \sigma(t), t \in I_k, k = 0, \ldots, m, \)
3. \( u(t) = g_k(t, u(t)) + \sigma_k, t \in I_k, k = 1, \ldots, m. \)
3. Existence of Solutions

We consider the following linear fractional differential equation

\[
\left(\frac{\rho D^{\alpha,\beta}_a u}{s_k^\gamma}\right)(t) = \psi(t), \quad t \in I_k, \ k = 0, \ldots, m,
\]

(12)

where \(0 < \alpha < 1\), \(0 \leq \beta \leq 1\), \(\rho > 0\), with the conditions

\[
u(t) = g_k(t, u(t)), \quad t \in \tilde{I}_k, \ k = 1, \ldots, m,
\]

(13)

\[
\left(\frac{\rho J^{1-\gamma}_a u}{s_k^\gamma}\right)(a^+) = \phi_0,
\]

(14)

where \(\gamma = \alpha + \beta - \alpha \beta\) and \(\phi_0 \in E\), and let \(\phi^* = \max\{\|\phi_k\| : k = 0, \ldots, m\}\). The following theorem shows that the problem (12)-(14) has a unique solution given by

\[
u(t) = \begin{cases}
\frac{\phi_k}{\Gamma(\gamma)} \left(\frac{t^\rho - s_k^\rho}{\rho}\right)^{\gamma-1} + \left(\frac{\rho J^{\alpha}_a \psi}{s_k^\gamma}\right)(t), & t \in I_k, \ k = 0, \ldots, m, \\
g_k(t, u(t)), & t \in \tilde{I}_k, \ k = 1, \ldots, m.
\end{cases}
\]

(15)

**Theorem 3.1.** Let \(\gamma = \alpha + \beta - \alpha \beta\), where \(0 < \alpha < 1\) and \(0 \leq \beta \leq 1\). If \(\psi : I_k \to E, k = 0, \ldots, m\), is a function such that \(\psi(\cdot) \in C(I_k, E)\), then \(u \in PC_{\gamma,\rho}(J)\) satisfies the problem (12)-(14) if and only if it satisfies (15).

**Proof.** Assume \(u\) satisfies (12)-(14). If \(t \in I_0\), then

\[
\left(\frac{\rho D^{\alpha,\beta}_a u}{s_k^\gamma}\right)(t) = \psi(t),
\]

Lemma 2.8 implies we have the solution can be written as

\[
u(t) = \frac{\left(\frac{\rho J^{1-\gamma}_a u}{s_k^\gamma}\right)(a)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \psi(s)ds.
\]

If \(t \in \tilde{I}_1\), then we have \(u(t) = g_1(t, u(t))\).

If \(t \in I_1\), then Lemma 2.8 implies

\[
u(t) = \frac{\left(\frac{\rho J^{1-\gamma}_a u}{s_k^\gamma}\right)(s_1)}{\Gamma(\gamma)} \left(\frac{t^\rho - s_1^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{s_1}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \psi(s)ds
\]

\[
= \frac{\phi_1}{\Gamma(\gamma)} \left(\frac{t^\rho - s_1^\rho}{\rho}\right)^{\gamma-1} + \left(\frac{\rho J^{\alpha}_a \psi}{s_k^\gamma}\right)(t).
\]

If \(t \in \tilde{I}_2\), then we have \(u(t) = g_2(t, u(t))\).

If \(t \in I_2\), then Lemma 2.8 implies

\[
u(t) = \frac{\left(\frac{\rho J^{1-\gamma}_a u}{s_k^\gamma}\right)(s_2)}{\Gamma(\gamma)} \left(\frac{t^\rho - s_2^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{s_2}^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \psi(s)ds
\]

\[
= \frac{\phi_2}{\Gamma(\gamma)} \left(\frac{t^\rho - s_2^\rho}{\rho}\right)^{\gamma-1} + \left(\frac{\rho J^{\alpha}_a \psi}{s_k^\gamma}\right)(t).
\]
Repeating the process in this way, the solution \( u(t) \) for \( t \in (a, b] \) can be written as

\[
u(t) = \begin{cases}
\frac{\phi_k}{\Gamma(\gamma)} \left( \frac{t^\gamma - s_k^\gamma}{\rho} \right) + \left( \rho \mathcal{J}_{s_k}^{\alpha} \psi \right)(t), & t \in I_k, \ k = 0, \ldots, m, \\
g_k(t, u(t)), & t \in \bar{I}_k, \ k = 1, \ldots, m.
\end{cases}
\]

Conversely, for \( t \in I_0 \), applying \( \rho \mathcal{J}_{s_k}^{\alpha - \gamma} \) on both sides of (15) and using Lemma 2.3 and Theorem 2.1 we get

\[
(\rho \mathcal{J}_{s_k}^{1 - \gamma} u)(t) = \phi_0 + (\rho \mathcal{J}_{s_k}^{1 - \gamma + \alpha} \psi)(t).
\]

Next, taking the limit as \( t \to a^+ \) of (16) and using Lemma 2.2 with \( 1 - \gamma < 1 - \gamma + \alpha \), we obtain

\[
(\rho \mathcal{J}_{s_k}^{1 - \gamma} u)(a^+) = \phi_0,
\]

which shows that the initial condition \( (\rho \mathcal{J}_{s_k}^{1 - \gamma} u)(a^+) = \phi_0 \), is satisfied. Next, for \( t \in I_k, k = 0, \ldots, m \), apply operator \( \rho \mathcal{D}_{s_k}^{\gamma} \) on both sides of (15). Then, from Lemma 2.3 and Lemma 2.6 we obtain

\[
(\rho \mathcal{D}_{s_k}^{\gamma} u)(t) = \left( \rho \mathcal{D}_{s_k}^{\beta(1-\alpha)} \psi \right)(t).
\]

Since \( u \in C_{\gamma, \rho}^1(I_k) \) and by definition of \( C_{\gamma, \rho}^1(I_k) \), we have \( \rho \mathcal{D}_{s_k}^{\gamma} u \in C_{\gamma, \rho}(I_k) \), and then (18) implies that

\[
(\rho \mathcal{D}_{s_k}^{\gamma} u)(t) = \left( \delta_0 \rho \mathcal{J}_{s_k}^{1-\beta(1-\alpha)} \psi \right)(t) = \left( \rho \mathcal{D}_{s_k}^{\beta(1-\alpha)} \psi \right)(t) \in C_{\gamma, \rho}(I_k).
\]

As \( \psi(\cdot) \in C(I_k, E) \) and from Lemma 2.1 it follows that

\[
(\rho \mathcal{J}_{s_k}^{1-\beta(1-\alpha)} \psi) \in C_{\gamma, \rho}(I_k), \ k = 0, \ldots, m.
\]

From (19), (20) and by the definition of the space \( C_{\gamma, \rho}^1(I_k) \), we obtain

\[
(\rho \mathcal{J}_{s_k}^{1-\beta(1-\alpha)} \psi) \in C_{\gamma, \rho}^1(I_k), \ k = 0, \ldots, m.
\]

Applying operator \( \rho \mathcal{J}_{s_k}^{\beta(1-\alpha)} \) on both sides of (18) and using Lemma 2.5, Lemma 2.2 and Property 2.2 we have

\[
\begin{align*}
(\rho \mathcal{D}_{s_k}^{\alpha, \beta} u)(t) &= \rho \mathcal{J}_{s_k}^{\beta(1-\alpha)} \left( \rho \mathcal{D}_{s_k}^{\gamma} u \right)(t) \\
&= \psi(t) - \left( \frac{\rho \mathcal{J}_{s_k}^{1-\beta(1-\alpha)} \psi}{\Gamma(\beta(1-\alpha))} \right) \frac{(\rho^\beta - s_k^\beta)^{\beta(1-\alpha)-1}}{\rho} = \psi(t),
\end{align*}
\]

that is, (12) holds.

Also, we have easily for \( u \in C(\bar{I}_k, E) \),

\[
u(t) = g_k(t, u(t_k)), \ t \in \bar{I}_k, \ k = 1, \ldots, m.
\]

This completes the proof. \(\square\)
As a consequence of Theorem 3.1 we have the following result:

**Lemma 3.1.** Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, and $k = 0, \ldots, m$, let $f : I_k \times E \times E \to E$, be a function such that $f(\cdot, u(\cdot), w(\cdot)) \in C(I_k, E)$, for any $u, w \in PC_{\gamma, \rho}(J)$. If $u \in PC_{\gamma, \rho}^{*}(J)$, then $u$ satisfies the problem $[1] - [3]$ if and only if $u$ is the fixed point of the operator $\Psi : PC_{\gamma, \rho}(J) \to PC_{\gamma, \rho}(J)$ defined by

$$
\Psi u(t) = \begin{cases}
\phi_k(t) \left( \frac{\|u - s_k^p\|}{\rho} \right)^{\gamma - 1} + \left( \rho^{\frac{\alpha}{s_k}} J_{s_k}^{\alpha} h \right)(t), & t \in I_k, \; k = 0, \ldots, m, \\
g_k(t, u(t)), & t \in \tilde{I}_k, \; k = 1, \ldots, m.
\end{cases}
$$

(21)

where $h \in C(I_k, E)$, $k = 0, \ldots, m$ is a function satisfying the functional equation

$$
h(t) = f(t, u(t), h(t)).
$$

Also, by Lemma 2.1 $\Psi u \in PC_{\gamma, \rho}(J)$.

The following hypotheses will be used in the sequel:

(Ax1) The function $t \mapsto f(t, u, w)$ is measurable on $I_k$, $k = 0, \ldots, m$, for each $u, w \in E$, and the functions $u \mapsto f(t, u, w)$ and $w \mapsto f(t, u, w)$ are continuous on $E$ for a.e. $t \in I_k$, $k = 0, \ldots, m$, and

$$
f(\cdot, u(\cdot), w(\cdot)) \in C_{\gamma, \rho}^{(1 - \alpha)}(I_k) \; \text{ for any } u, w \in PC_{\gamma, \rho}(J).
$$

(Ax2) There exists a continuous function $p : [a, b] \to [0, \infty)$ such that

$$
\|f(t, u, w)\| \leq p(t), \; \text{for a.e. } t \in I_k, \; k = 0, \ldots, m, \; \text{and for each } u, w \in E.
$$

(Ax3) For each bounded set $B \subset E$ and for each $t \in I_k$, $k = 0, \ldots, m$, we have

$$
\mu(f(t, B, (\rho^\alpha D_{s_k}^{\alpha, \beta}) B)) \leq p(t) \mu(B),
$$

where $\rho^\alpha D_{s_k}^{\alpha, \beta} B = \{\rho^\alpha D_{s_k}^{\alpha, \beta} w : w \in B\}$.

(Ax4) The functions $g_k \in C(\bar{I}_k, E)$, $k = 1, \ldots, m$, and there exists $l^* > 0$ such that

$$
\|g_k(t, u)\| \leq l^* \|u\| \; \text{ for each } u \in E, \; k = 1, \ldots, m.
$$

(Ax5) For each bounded set $B \subset E$ and for each $t \in \bar{I}_k$, $k = 1, \ldots, m$, we have

$$
\mu(g_k(t, B)) \leq l^* \mu(B), \; k = 1, \ldots, m.
$$

Set $p^* = \sup_{t \in [a, b]} p(t)$.

We are now in a position to state and prove our existence result for the problem $[1] - [3]$ based on Mönch’s fixed point theorem.

**Theorem 3.2.** Assume (Ax1)-(Ax5) hold. If

$$
L := \max \left\{ l^*, \frac{p^* (\gamma)}{1 + (\gamma)} \left( \frac{w - a \rho}{\rho} \right)^\alpha \right\} < 1,
$$

(22)

then the problem $[1] - [3]$ has at least one solution in $PC_{\gamma, \rho}(J)$. 

Proof. Consider the operator $\Psi : PC_{\gamma,\rho}(J) \to PC_{\gamma,\rho}(J)$ defined in (21) and the ball $B_R := B(0,R) = \{ w \in PC_{\gamma,\rho}(J) : \|w\|_{PC_{\gamma,\rho}} \leq R \}$, such that

$$R \geq \frac{\phi^*}{(1-t^*)\Gamma(\gamma)} + \frac{p^*}{(1-t^*)\Gamma(\alpha + 1)} \left( \frac{b^\rho-a^\rho}{\rho} \right)^{1-\gamma + \alpha}. $$

For any $u \in B_R$, and each $t \in I_k$, $k = 0, \ldots, m$, we have

$$\|\Psi u(t)\| \leq \frac{\|\phi_k\|}{\Gamma(\gamma)} \left( \frac{t^\rho-s_k^\rho}{\rho} \right)^{\gamma-1} + \left( p^\alpha \frac{\|h(s)\|}{s_k^\rho} \right) (t) \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha + 1)} \left( \frac{t^\rho-s_k^\rho}{\rho} \right)^{\gamma-1} + p^* \left( p^\alpha \frac{\|h(s)\|}{s_k^\rho} \right) (t).$$

By Lemma 2.3, we have

$$\left\| \left( \frac{t^\rho-s_k^\rho}{\rho} \right)^{1-\gamma} \Psi u(t) \right\| \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha + 1)} \left( \frac{t^\rho-s_k^\rho}{\rho} \right)^{1-\gamma + \alpha} \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha + 1)} \left( \frac{b^\rho-a^\rho}{\rho} \right)^{1-\gamma + \alpha}. $$

And for $t \in \tilde{I}_k$, $k = 1, \ldots, m$, we have

$$\| (\Psi u)(t) \| \leq l^*\|u(t)\| \leq l^* R. $$

Hence,

$$\|\Psi u\|_{PC_{\gamma,\rho}} \leq l^* R + \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^*}{\Gamma(\alpha + 1)} \left( \frac{b^\rho-a^\rho}{\rho} \right)^{1-\gamma + \alpha} \leq R.$$ 

This proves that $\Psi$ transforms the ball $B_R$ into itself. We shall show that the operator $\Psi : B_R \to B_R$ satisfies all the assumptions of Theorem 2.4. The rest of the proof will be given in several steps.

**Step 1:** $\Psi : B_R \to B_R$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC_{\gamma,\rho}(J)$. Then for each $t \in I_k$, $k = 0, \ldots, m$, we have,

$$\left\| \left( (\Psi u_n)(t) - (\Psi u)(t) \right) \left( \frac{t^\rho-s_k^\rho}{\rho} \right)^{1-\gamma} \right\| \leq \left( \frac{t^\rho-s_k^\rho}{\rho} \right)^{1-\gamma} \left( p^\alpha \frac{\|h_n(s) - h(s)\|}{s_k^\rho} \right) (t),$$

where $h_n, h \in C(I_k, E); k = 0, \ldots, m$, such that

$$h_n(t) = f(t, u_n(t), h_n(t)), \quad h(t) = f(t, u(t), h(t)).$$

For each $t \in \tilde{I}_k$, $k = 1, \ldots, m$, we have,

$$\|((\Psi u_n)(t) - (\Psi u)(t))\| \leq \|(g_k(t, u_n(t)) - g_k(t, u(t)))\|. $$

Since $u_n \to u$, then we get $h_n(t) \to h(t)$ as $n \to \infty$ for each $t \in (a,b]$, and since $f$ and $g_k$ are continuous, then we have

$$\|\Psi u_n - \Psi u\|_{PC_{\gamma,\rho}} \to 0 \text{ as } n \to \infty.$$ 

**Step 2:** $\Psi(B_R)$ is bounded and equicontinuous.

Since $\Psi(B_R) \subset B_R$ and $B_R$ is bounded, then $\Psi(B_R)$ is bounded.
Next, let $\epsilon_1, \epsilon_2 \in I_k, k = 0, \ldots, m$, $\epsilon_1 < \epsilon_2$, and let $u \in B_R$. Then
\[
\left\| \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} (\Psi u)(\epsilon_1) - \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} (\Psi u)(\epsilon_2) \right\| 
\leq \left\| \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \rho \int_{s_k}^{\bar{s}} h(\tau) \right)(\epsilon_1) - \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \rho \int_{s_k}^{\bar{s}} h(\tau) \right)(\epsilon_2) \right\|
\leq \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \rho \int_{s_k}^{\bar{s}} \|h(\tau)\| \right) (\epsilon_2) + \frac{1}{\Gamma(\alpha)} \int_{s_k}^{\bar{s}} \|\tau^{\rho - 1} H(\tau)h(\tau)\| \, d\tau,
\]

where $H(\tau) = \left[ \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{\alpha - 1} - \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{\alpha - 1} \right]$.

Then by Lemma 2.3, we have
\[
\left\| \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} (\Psi u)(\epsilon_1) - \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} (\Psi u)(\epsilon_2) \right\| 
\leq \frac{p^\alpha}{\Gamma(1 + \alpha)} \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{\alpha} + p^\alpha \int_{s_k}^{\bar{s}} \left\| \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{\alpha} - \right\| \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{1 - \gamma} \left( \frac{\epsilon_1^\alpha - \epsilon_2^\alpha}{\rho} \right)^{\alpha} \right\| \, d\tau,
\]

and for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have
\[
\| (\Psi u)(\epsilon_1) - (\Psi u)(\epsilon_2) \| \leq \| (g_k(\epsilon_1, u(\epsilon_1))) - (g_k(\epsilon_2, u(\epsilon_2))) \|.
\]

As $\epsilon_1 \to \epsilon_2$, the right-hand side of the above inequality tends to zero. Hence, $\Psi(B_R)$ is bounded and equicontinuous.

**Step 3:** The implication (i) of Theorem 2.4 holds. Now let $D$ be an equicontinuous subset of $B_R$ such that $D \subset \overline{\Psi(D)} \cup \{0\}$, therefore the function $t \to d(t) = \mu(D(t))$ are continuous on $J$. By (A2), (A3) and the properties of the measure $\mu$, for each $t \in I_k, k = 0, \ldots, m$, we have
\[
\left( \frac{\epsilon_r - s_r}{\rho} \right)^{1 - \gamma} d(t) \leq \mu \left( \left( \frac{\epsilon_r - s_r}{\rho} \right)^{1 - \gamma} (\Psi D)(t) \cup \{0\} \right)
\leq \mu \left( \left( \frac{\epsilon_r - s_r}{\rho} \right)^{1 - \gamma} (\Psi D)(t) \right)
\leq \left( \frac{\epsilon_r - s_r}{\rho} \right)^{1 - \gamma} \left( \rho \int_{s_k}^{\bar{s}} s(\tau) \mu(D(s)) \right) (t)
\leq p^\alpha \left( \frac{\epsilon_r - s_r}{\rho} \right)^{1 - \gamma} \left( \rho \int_{s_k}^{\bar{s}} d(s) \right) (t)
\leq \left[ \frac{p^\alpha}{\Gamma(1 + \alpha)} \left( \frac{\epsilon_r - s_r}{\rho} \right)^{1 - \gamma} \right] \|d\| \|PC_{\gamma,\rho} \|
\]

And for each $t \in \tilde{I}_k, k = 1, \ldots, m$, we have
\[
d(t) \leq \mu (g_k(t, D(t))) \leq l^* d(t).
\]

Thus for each $t \in (a, b]$, we have
\[
\|d\| \|PC_{\gamma,\rho} \| \leq L \|d\| \|PC_{\gamma,\rho} \|
\]

From (22), we get $\|d\| \|PC_{\gamma,\rho} \| = 0$, that is, $d(t) = \mu(D(t)) = 0$, for each $t \in (a, b]$, and then $D(t)$ is relatively compact in $E$. In view of the Ascoli-Arzelà Theorem, $D$ is relatively compact in $B_R$. Applying now Theorem 2.4, we conclude that $\Psi$ has a fixed point $u^* \in PC_{\gamma,\rho}(J)$, which is solution of the problem (1)-(3).

**Step 4:** We show that such a fixed point $u^* \in PC_{\gamma,\rho}(J)$ is actually in $PC_{\gamma,\rho}(J)$.

Since $u^*$ is the unique fixed point of operator $\Psi$ in $PC_{\gamma,\rho}(J)$, then for each $t \in J$, we have
\[
\Psi u^*(t) = \begin{cases}
\phi_k \left( \frac{\epsilon_r - s_r}{\rho} \right)^{1 - \gamma} \left( \rho \int_{s_k}^{\bar{s}} h(\tau) \right) (t), & t \in I_k, k = 0, \ldots, m, \\
g_k(t, u^*(t)), & t \in \tilde{I}_k, k = 1, \ldots, m.
\end{cases}
\]
where \( h \in C(I_k, E); k = 0, \ldots, m \), such that

\[
h(t) = f(t, u^*(t), h(t)).
\]

For \( t \in I_k; k = 0, \ldots, m \), applying \( \varphi D_{s_k}^{\gamma} \) to both sides and by Lemma 2.3 and Lemma 2.6 we have

\[
\varphi D_{s_k}^{\gamma} u^*(t) = \left( \varphi D_{s_k}^{\gamma} \rho J_{s_k}^{\alpha} f(s, u^*(s), h(s)) \right)(t).
\]

Hence, for each \( t \), we deduce that \( \Psi \) is an \( L \)-contraction. As consequence of Theorem 2.5 and using Step 4 of the last result, we deduce that \( \Psi \) has a fixed point which is a solution of the problem (1)–(3).

Our second existence result for the problem (1)–(3) is based on Darbo’s fixed point theorem.

**Theorem 3.3.** Assume (Ax1)–(Ax5) hold. If

\[
L := \max \left\{ t^*, \frac{\mu^* \Gamma(\gamma)}{\Gamma(\alpha + \gamma)} \left( \frac{b^0 - a^0}{h} \right)^{\alpha} \right\} < 1,
\]

then the problem (1)–(3) has at least one solution in \( PC_{\gamma, \rho}(I) \).

**Proof.** Consider the operator \( \Psi \) defined in (21). We know that \( \Psi : B_R \rightarrow B_R \) is bounded and continuous and that \( \Psi(B_R) \) is equicontinuous. We need to prove that the operator \( \Psi \) is an \( L \)-contraction. Let \( D \subset B_R \) and \( t \in I_k, k = 0, \ldots, m \). Then we have

\[
\mu \left( \frac{t^0 - s_k^0}{h} \right)^{1-\gamma} (\Psi D)(t) \leq \mu \left( \frac{t^0 - s_k^0}{h} \right)^{1-\gamma} (\Psi u)(t) : u \in D
\]

\[
\leq \left( \frac{b^0 - a^0}{h} \right)^{1-\gamma} \left\{ \rho J_{s_k}^{\alpha} \mu(u(s)) \right\} (t), u \in D\right\}.
\]

By Lemma 2.3, we have for \( t \in I_k, k = 0, \ldots, m, \)

\[
\mu \left( \frac{t^0 - s_k^0}{h} \right)^{1-\gamma} (\Psi D)(t) \leq \left[ \frac{\mu^* \Gamma(\gamma)}{\Gamma(\alpha + \gamma)} \left( \frac{b^0 - a^0}{h} \right)^{\alpha} \right] \mu_{PC_{\gamma, \rho}}(D).
\]

And for each \( t \in I_k, k = 1, \ldots, m \), we have

\[
\mu ((\Psi D)(t)) \leq \mu (g_k(t, D(t))) \leq l^* \mu(D(t)).
\]

Hence, for each \( t \in (a, b] \), we have

\[
\mu_{PC_{\gamma, \rho}}(\Psi D) \leq L \mu_{PC_{\gamma, \rho}}(D).
\]

So, by (22), the operator \( \Psi \) is an \( L \)-contraction. As consequence of Theorem 2.5 and using Step 4 of the last result, we deduce that \( \Psi \) has a fixed point which is a solution of the problem (1)–(3).


First, we are concerned with the Ulam-Hyers-Rassias (U-H-R) stability of our problem (1)–(3).

**Theorem 4.1.** Assume that in addition to (Ax1)–(Ax5) and (22), the following hypotheses hold.
(Ax6) There exist a nondecreasing function \( \vartheta : (a, b) \rightarrow [0, \infty) \) and \( \lambda_0 > 0 \) such that for each \( t \in I_k, k = 0, \ldots, m \), we have
\[
(\rho J^\alpha_{s_k} \vartheta)(t) \leq \lambda_0 \vartheta(t).
\]

(Ax7) There exists a continuous function \( \chi : \bigcup_{k=1}^m [s_k, t_{k+1}] \rightarrow [0, \infty) \) such that for each \( t \in I_k, k = 0, \ldots, m \), we have
\[
p(t) \leq \chi(t) \vartheta(t).
\]

Then problem (1)-(3) is U-H-R stable with respect to \((\vartheta, \tau)\).

Proof. Consider the operator \( \Psi \) defined in (21). Let \( u \in PC_{\gamma, \rho}(J) \) be a solution of inequality (11), and let us assume that \( w \) is the unique solution of the problem
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\left( \rho D^{\alpha, \beta}_{s_k} w \right)(t) = f(t, w(t), \left( \rho D^{\alpha, \beta}_{s_k} w \right)(t)) ; t \in I_k, k = 0, \ldots, m, \\
w(t) = g_k(t, w(t)) ; t \in \bar{I}_k, k = 1, \ldots, m,
\end{array} \right.
\end{aligned}
\]
\[
\begin{aligned}
&\left\{ \begin{array}{l}
\left( \rho J^{\gamma, \gamma}_{s_k} w \right)(s_k^+) = \left( \rho J^{\gamma, \gamma}_{s_k} u \right)(s_k^+) = \phi_k, k = 0, \ldots, m.
\end{array} \right.
\end{aligned}
\]

By Lemma 3.1 we obtain for each \( t \in (a, b) \)
\[
w(t) = \left\{ \begin{array}{ll}
\frac{\phi_k}{\Gamma(\gamma)} \left( \frac{\rho - s_k}{\rho} \right)^{\gamma-1} + \left( \rho J^\alpha_{s_k} h \right)(t) & \text{if } t \in I_k, k = 0, \ldots, m, \\
g_k(t, w(t)) & \text{if } t \in \bar{I}_k, k = 1, \ldots, m,
\end{array} \right.
\]
where \( h \in C(I_k, E), k = 0, \ldots, m, \) is a function satisfying the functional equation
\[
h(t) = f(t, w(t), h(t)).
\]

Since \( u \) is a solution of the inequality (11), by Remark 2.1 we have
\[
\left\{ \begin{array}{l}
\left( \rho D^{\alpha, \beta}_{s_k} u \right)(t) = f(t, u(t), \left( \rho D^{\alpha, \beta}_{s_k} u \right)(t)) + \sigma(t), t \in I_k, k = 0, \ldots, m, \\
u(t) = g_k(t, u(t)) + \sigma_k, t \in I_k, k = 1, \ldots, m.
\end{array} \right.
\]  \hspace{1cm} (23)

Clearly, the solution of (23) is given by
\[
u(t) = \left\{ \begin{array}{ll}
\frac{\phi_k}{\Gamma(\gamma)} \left( \frac{\rho - s_k}{\rho} \right)^{\gamma-1} + \left( \rho J^\alpha_{s_k} g \right)(t) + \left( \rho J^\alpha_{s_k} \sigma \right)(t), & t \in I_k, k = 1, \ldots, m, \\
g_k(t, u(t)) + \sigma_k, & t \in \bar{I}_k, k = 1, \ldots, m,
\end{array} \right.
\]
where \( g : I_k \rightarrow E, k = 0, \ldots, m, \) is a function satisfying the functional equation
\[
g(t) = f(t, u(t), g(t)).
\]

Hence, for each \( t \in I_k, k = 0, \ldots, m, \) we have
\[
\|u(t) - w(t)\| \leq \left( \rho J^\alpha_{a^+} \|g(s) - h(s)\| \right)(t) + \left( \rho J^\alpha_{a^+} \|\sigma(s)\| \right)(t)
\]
\[
\leq \epsilon \lambda_0 \vartheta(t) + \int_a^t s^{\rho-1} \left( \frac{\rho - s_k}{\rho} \right)^{\alpha-1} 2x(t) \vartheta(t) ds
\]
\[
\leq \epsilon \lambda_0 \vartheta(t) + 2\chi^* \left( \rho J^\alpha_{a^+} \vartheta \right)(t)
\]
\[
\leq \left( \epsilon + 2\chi^* \right) \lambda_0 \vartheta(t)
\]
\[
\leq \left( 1 + \frac{2\chi^*}{\epsilon} \right) \lambda_0 \epsilon (\tau + \vartheta(t)),
\]
where
\[ \chi^* = \max_{k=0, \ldots, m} \left\{ \sup_{t \in [s_k, t_{k+1}]} \chi(t) \right\}. \]

For each \( t \in \tilde{I}_k, k = 1, \ldots, m \), we have
\[ \|u(t) - w(t)\| \leq \|g_k(t, u(t)) - g_k(t, w(t))\| + \|\sigma_k\| \leq \|u(t) - w(t)\| + \epsilon \tau, \]
and then by [22],
\[ \|u(t) - w(t)\| \leq \frac{\epsilon \tau}{1 - \tau} \leq \frac{\epsilon \tau}{1 - \tau} (\sigma + \vartheta(t)). \]

Then for each \( t \in (a, b], \) we have
\[ \|u(t) - w(t)\| \leq a_\vartheta \epsilon (\tau + \vartheta(t)), \]
where
\[ a_\vartheta = \max \left\{ \frac{1}{\epsilon} \chi^* \vartheta, \frac{1}{1 - \lambda \vartheta} \right\}. \]

Hence, problem (1)–(3) is U-H-R stable with respect to \((\vartheta, \tau)\). \( \square \)

5. Example

Let 
\[ E = l^1 = \left\{ v = (v_1, v_2, \ldots, v_n, \ldots), \sum_{n=1}^{\infty} |v_n| < \infty \right\} \]
be the Banach space with the norm
\[ \|v\| = \sum_{n=1}^{\infty} |v_n|. \]

Consider the following initial value problem with non-instantaneous impulses
\[ \left( \frac{1}{s_k} D^{\frac{1}{s_k}} u \right)(t) = f \left( t, u(t), \frac{1}{s_k} D^{\frac{1}{s_k}} u \right)(t) \right), t \in (1, 2] \cup \{e, 3\}, k \in \{0, 1\} \]
\[ u(t) = g(t, u(t)), \quad t \in (2, e], \]
\[ \left( \frac{1}{s_k} D^{\frac{1}{s_k}} u \right)(1^+) = 0, \]
where
\[ a = t_0 = s_0 = 1 < t_1 = 2 < s_1 = e < t_2 = 3 = b, \]
\[ u = (u_1, u_2, \ldots, u_n, \ldots), \]
\[ f = (f_1, f_2, \ldots, f_n, \ldots), \]
\[ \frac{1}{s_k} D^{\frac{1}{s_k}} u = \left( \frac{1}{s_k} D^{\frac{1}{s_k}} u_1, \ldots, \frac{1}{s_k} D^{\frac{1}{s_k}} u_2, \ldots, \frac{1}{s_k} D^{\frac{1}{s_k}} u_n, \ldots, \right) , \]
\[ g = (g_1, g_2, \ldots, g_n, \ldots) , \]
\[ f_n(t, u_n(t), \left( \frac{1}{s_k} D^{\frac{1}{s_k}} u_n \right)(t)) = \frac{(2t^3 + 5e^{-2})|u_n(t)|}{183e^{-t+3}(1 + \|u(t)\| + \| \left( \frac{1}{s_k} D^{\frac{1}{s_k}} u \right)(t) \|)} , \]
with \( t \in (1, 2] \cup \{e, 3\}, k \in \{0, 1\}, n \in \mathbb{N}, \) and
\[ g_n(t, u_n(t)) = \frac{|u_n(t)|}{105e^{t+5} + 1}, \quad t \in (2, e], n \in \mathbb{N}. \]
We have
\[ C_{γ,ρ}^{1−α}((1,2)) = C_{1,1}^0((1,2)) = \{ h : (1,2] \to E : (\sqrt{t−1})h \in C((1,2],E) \}, \]
and
\[ C_{γ,ρ}^{1−α}((e,3)) = C_{1,1}^0((e,3)) = \{ h : (e,3] \to E : (\sqrt{t−e})h \in C([e,3],E) \}, \]
with \( γ = \frac{1}{2}, \rho = 1, \beta = 0 \) and \( k \in \{0,1\} \). Clearly, the continuous function \( f \in C_{1,1}^0((1,2]) \cup C_{1,1}^0((e,3]). \)
Hence the condition (Ax1) is satisfied.
For each \( u, w \in E \) and \( t \in (1,2] \cup (e,3], \)
\[ \|f(t,u,w)\| \leq \frac{2t^3 + 5e^{-2}}{183e^{-t+3}}. \]
Hence condition (Ax2) is satisfied with
\[ p(t) = \frac{2t^3 + 5e^{-2}}{183e^{-t+3}}, \]
and
\[ p^∗ = \frac{54 + 5e^{-2}}{183}. \]
And for each \( u \in E \) and \( t \in (2,e] \) we have
\[ \|g(t,u)\| \leq \frac{\|u\|}{105e^{t−e}+1}, \]
and so the condition (Ax4) is satisfied with \( l^∗ = \frac{1}{105e^{t−e}+1}. \)
The condition (22) of Theorem 3.2 is satisfied, for
\[ L := \max \left\{ l^∗, \frac{p^∗Γ(γ)}{Γ(α+γ)} \left( \frac{b^α−a^α}{ρ} \right)^α \right\} \approx 0.7489295248 < 1. \]
Let \( Ω \) be a bounded set in \( E \) where \( ^1D_{s_k}^{1,0}Ω = \left\{ ^1D_{s_k}^{1,0}v : v \in Ω \right\}, k \in \{0,1\} \). Then by the properties of the Kuratowski measure of noncompactness, for each \( u \in Ω \) and \( t \in (1,2] \cup (e,3], \) we have
\[ μ\left( f(t,Ω,^1D_{s_k}^{1,0}Ω) \right) \leq p(t)μ(Ω), \]
and for each \( t \in (2,e], \)
\[ μ(\{ g(t,Ω) \}) \leq l^∗μ(Ω). \]
Hence conditions (Ax3) and (Ax5) are satisfied. Then the problem (24)–(26) has at least one solution in \( PC_{1,1}((1,3]). \)
Also, hypothesis (Ax8) is satisfied with \( τ = 1 \) and
\[ θ(t) = \begin{cases} \frac{1}{\sqrt{t−s_k}}, & t \in (1,2] \cup (e,3], \\ 1, & t \in (2,e], \end{cases} \]
and \( λ_θ = \sqrt{π}. \) Indeed, for each \( t \in (1,2], \) we get
\[ (^1J_{1,t}^{1}θ)(t) = \sqrt{π} ≤ \frac{\sqrt{π}}{\sqrt{t−1}}, \]
and for each \( t \in (e,3], \) we get
\[ (^1J_{e,t}^{1}θ)(t) = \sqrt{π} ≤ \frac{\sqrt{π}}{\sqrt{t−e}}. \]
Let the function $\chi : [1, 2] \cup [e, 3] \rightarrow [0, \infty)$ be defined by

$$
\chi(t) = \frac{(2t^3 + 5e^{-2})\sqrt{t - s_k}}{183e^{-t+3}}, k \in \{0, 1\}.
$$

Then, for each $t \in (1, 2] \cup (e, 3]$, we have

$$p(t) = \chi(t)d(t),$$

with $\chi^* = p^*$. Hence, the condition $(Ax9)$ is satisfied. Consequently, Theorem 4.1 implies that the problem $(24)-(26)$ is U-H-R stable.

References


