Comment on strongly preirresolute topological vector spaces

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Abstract

A subset $A$ of a topological space $X$ is said to be pre-open if $A \subseteq Int(Cl(A))$. Let $PO(X)$ denote the family of all pre-open sets in a given topological space $X$. In general, $PO(X)$ does not form a topology on $X$. Furthermore, in topological vector spaces, it is not always true that $PO(L)$ forms a topology on $L$ when $L$ is a topological vector space. In this note, we prove that the class of strongly preirresolute topological vector spaces is that subclass of topological vector spaces in which $PO(L)$ forms a topology and thereby we will observe that all results which are proven in [5] concerning strongly preirresolute topological vector spaces are obvious.

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1. Introduction and the main result

Let $(X, \mathcal{O})$ (or simply, $X$) be a topological space. A subset $A \subseteq X$ is called pre-open if $A \subseteq Int(Cl(A))$. The complement of a pre-open set is called pre-closed set. Let $PO(X)$ denote the collection of all pre-open subsets of $X$. It is well-known that in general, $PO(X)$ does not form a topology on $X$. Furthermore, consider a topological vector space $L = \mathbb{R}$, where $\mathbb{R}$ is endowed with the standard topology. Now,

let $A = \{x \in \mathbb{Q}: 0 < x < 1\}$ and $B = \{x \in \mathbb{R}: x \notin \mathbb{Q}, 0 < x < 1\} \cup \{\frac{1}{2}\}$ where $\mathbb{Q}$ denotes the set of rational numbers.

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Obviously, both $A$ and $B$ are pre-open subsets of $\mathbb{R}$ but $A \cap B = \{ \frac{1}{2} \}$ is not pre-open. Thus, we have seen that in topological vector spaces, $PO(L)$ need not form a topology on $L$ when $L$ is a topological vector space.

**Definition 1.1.** Let $X$, $Y$ be two topological spaces. A function $f : X \rightarrow Y$ is called $p$–continuous if the inverse image of any pre-open subset of $Y$ is open in $X$.

**Definition 1.2.** A topological space $X$ is called pre-$T_2$ [3] if for each pair of distinct points $x$ and $y$ in $X$, there exist disjoint pre-open sets $U$ and $V$ of $X$ such that $x \in U$ and $y \in V$.

**Definition 1.3.** A subset $A$ of a topological space $X$ is called strongly compact [4] if every cover of $A$ by pre-open sets in $X$ has a finite subcover.

In [5], Rajesh and Vijayabharathi (2013) introduced the notion of strongly preirresolute topological vector spaces and established several results in strongly preirresolute topological vector spaces.

A pair $(L, \mathcal{F})$ (or simply, $L$) is called a strongly preirresolute topological vector space if:

- $L$ is a real vector space, and
- $\mathcal{F}$ is a topology on $L$ such that the vector space operations are $p$–continuous.

In fact, this definition can be extended to all complex vector spaces like topological vector spaces. Evidently, every strongly preirresolute topological vector space is a topological vector space but the converse is not true, in general because $(\mathbb{R}, \mathcal{F})$ is not a strongly preirresolute topological vector space.

This note concerns the paper [5] by Rajesh and Vijayabharathi. We exhibit that all theorems in [5] are the particular cases of well-known results of topological vector spaces which follow directly from the following fact:

**Theorem 1.1.** Let $(L, \mathcal{F})$ be a strongly preirresolute topological vector space. Then $PO(L)$ forms a topology on $L$.

**Proof.** To prove this theorem, it is enough to show that every pre-open set of $L$ is open. For, let $A$ be any pre-open set of $L$ and let $x \in A$ be any element.

Since the vector addition mapping of cartesian product $L \times L$ into $L$ is $p$–continuous, there exist open sets $U$ of $L$ containing $0$ and $V$ of $L$ containing $x$ such that $U + V \subseteq A$. In particular, $0 + V \subseteq A$. This indicates that $x$ is an interior point of $A$. Thus, $A$ is open. Hence $PO(L) = \mathcal{F}$.

**Corollary 1.1.** If $(L, \mathcal{F})$ is a strongly preirresolute topological vector space, then we have

1. A subset $A \subseteq L$ is strongly compact if and only if it is compact.
2. $(L, \mathcal{F})$ is pre-$T_2$ space if and only if it is $T_2$ space.

**Remark 1.1.** All results (for example, Theorem 3.9, Theorem 3.11, Theorem 3.13 and Theorem 3.18) in [5] follow directly by Corollary 1.1.1 together with corresponding well-known results in topological vector spaces (for example, see [2, Proposition 2.2.3, Corollary 2.2.4], [6, Theorem 1.10] and [7]).

We now formulate an alternative definition of Hahn Banach Separation Theorem in strongly preirresolute topological vector spaces.

**Theorem 1.2.** Suppose $A$, $B$ are disjoint, non-empty convex sets in a strongly preirresolute topological vector space $L$.

(a) If $A$ is pre-open, then there is a linear continuous map $\varphi : L \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$ s.t. $\mathop{\sup}{\{Re \varphi(x) : x \in A\}} < Re \varphi(y)$, for all $y \in B$.

(b) If $B$ is strongly compact, $A$ pre-closed, and $L$ is locally convex, then there is a linear continuous map $\varphi : L \rightarrow \mathbb{R}$, $\lambda \in \mathbb{R}$ and $\epsilon > 0$ s.t. $\forall x \in B$, $y \in A$, $Re \varphi(x) < \lambda < \lambda + \epsilon < Re \varphi(y)$.

**Proof.** Follows from Theorem 1.4 and [1, Theorem 5.7].
References